

**Dynamics of D-branes I. The non-Abelian Dirac-Born-Infeld action,
its first variation, and the equations of motion for D-branes
— with remarks on the non-Abelian Chern-Simons/Wess-Zumino term**

Chien-Hao Liu and Shing-Tung Yau

Abstract

In earlier works, D(1) (arXiv:0709.1515 [math.AG]), D(11.1) (arXiv:1406.0929 [math.DG]), D(11.2) (arXiv:1412.0771 [hep-th]), and D(11.3.1) (arXiv:1508.02347 [math.DG]), we have explained, and shown by feature stringy examples, why a D-brane in string theory, when treated as a fundamental dynamical object, can be described by a map φ from an Azumaya/matrix manifold X^{A_z} (cf. the D-brane world-volume) with a fundamental module with a connection (E, ∇) (cf. the Chan-Paton bundle) to the target space-time Y . In this sequel, we construct a non-Abelian Dirac-Born-Infeld action functional $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$ for such pairs (φ, ∇) when the target space-time Y is equipped with a background (dilaton, metric, B)-field (Φ, g, B) from closed strings. We next develop a technical tool needed to study variations of this action and apply it to derive the first variation $\delta S_{DBI}^{(\Phi, g, B)}/\delta(\varphi, \nabla)$ of $S_{DBI}^{(\Phi, g, B)}$ with respect to (φ, ∇) . The equations of motion that govern the dynamics of D-branes then follow. A complete action for a D-brane world-volume must include also the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C)}(\varphi, \nabla)$ that governs how the D-brane world-volume couples with the Ramond-Ramond fields C on Y . In this work, a version $S_{CS/WZ}^{(C, B)}(\varphi, \nabla)$ of non-Abelian Chern-Simons/Wess-Zumino action functional for (φ, ∇) that follows the same guide with which we construct $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$ is constructed for lower-dimensional D-branes (i.e. D(-1)-, D0-, D1-, D2-branes). Its first variation $\delta S_{CS/WZ}^{(C, B)}(\varphi, \nabla)/\delta(\varphi, \nabla)$ is derived and its contribution to the equations of motion for (φ, ∇) follows. For D-branes of dimension ≥ 3 , an anomaly issue needs to be understood in the current context. The current notes lay down a foundation toward the dynamics of D-branes along the line of this D-project.

Key words: D-brane, Azumaya/matrix manifold, Dirac-Born-Infeld action, Chern-Simons/Wess-Zumino term, ring-homomorphism, higher-order derivation, first variation, Euler-Lagrange equation of motion

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*Chien-Hao Liu dedicates the current notes to Ling-Miao Chou
for her tremendous love that makes this work/project possible,
and to his parents-in-law Mr. & Mrs. Shih-Chuan Chow (1919–2011) and Min-Chih Liu,
who handed over him their most precious gem*.*

*(From C.H.L.) ON THE ROAD

When I first met my then father-in-law-to-be, he had already retired for a decade. Like many Chinese around his generation, he grew up in a chaotic China and went to schools between civil wars and the invasion of Japan. After his graduation from Wuhan University through a government aid, majoring in electrical engineering, he served three years at the power plant at Yibin, Sichuan. After World War II, he got scholarships first from the then Central Government of China and later from the United Aid to China Fund, London, to work study in Scotland, England. That was a time when a trip between Sichuan, China, and Southampton, England, took about a month, including a stop at India, mixing airplanes, railroads, and ships. Seven years afterwards, when he was about to go home, he faced one of the most difficult decisions he had to make: Mainland China or Taiwan, since China had been divided into two political entities after the four years civil war following the surrender of Japan. Whatever reason behind his decision, he chose the latter and became a member of the team that constructed the modern power plants and power system at Taiwan. Only decades later in late 1980s when the Taiwan government and the Mainland China government started to build up a reconciling atmosphere, he got a chance to go back to see his siblings again, though his parents had long passed away. That is a tragedy many in his generation underwent. Also like many in his generation who had seen and experienced in person enough sufferings but had the luck to get to finish high-level education, he took his service and devotion to the common good of his country as a responsibility without leaving a name behind. This is a very brief story of Mr. Chow, who started his own family quite late and, for that reason, held his three children, with Ling-Miao the youngest and only daughter, very dearly. Like Prof. Raoul Bott, he maintained a curious mind even to his senior years and did a few calculus exercises(!!!) daily to keep his mind active and alert.

Fast forward to October 2015, completely unforeseeable while in the early summer that year, I was once again on the road, repeating the same route from Austin, Texas, to Boston, Massachusetts, I'd followed more than a decade ago with Ling-Miao. This is the fourth time I made such a trip on the road across the United States and, as in the case of my father-in-law, each such trip marks a huge transition in my study and life. The first time was in early 1990s from Princeton, New Jersey, to Berkeley, California. I felt like heading toward a new world. That was the summer a year after I had met Ling-Miao for the first time unexpectedly in a trip to Ohio without knowing that she would be influencing my life forever. The second time was in mid-1990s from Coral Gable, Florida, to Austin, Texas. That marked the official end of my student years. The third time was at the turn of the millennium from Austin, Texas, to Cambridge, Massachusetts. That is the only road-trip I ever made with Ling-Miao. The fall before that trip I was once again in the job market. I got a surprise contact near the Thanksgiving holiday from Prof. Brian Greene at Columbia concerning a position in his group based on an earlier work I had done ('*On the isolated singularity of a 7-space obtained by rolling Calabi-Yau threefolds through extremal transitions*', arXiv:hep-th/9801175v1; revised in v2 with Volker Braun in Candelas' group then at U.T. Austin) and the recommendation from Prof.'s Orlando Alvarez, Jacques Distler, and Daniel Freed. One or two weeks later disappointing follow-up news came from Prof. Greene: That position he had kindly intended to offer requires a U.S. Citizenship, which I hadn't acquired at that time. Ordinarily, an apology as a formality is more than enough to end this contact since it is not his fault at all that it didn't work out. Yet, this is what Prof. Greene showed his nobleness and generosity: Rather than just ending with that, he recommended me further to Prof. Yau at Harvard. Thus, thanks to this unexpected twist of events and Prof. Yau's acceptance, I came to Harvard in pure luck as an even bigger surprise.

The two leading institutes at the Boston area together provide a unique soil for a curious and absorbing mind: the frontier research and related basic and/or topic courses in algebraic geometry, differential geometry, and symplectic geometry on the mathematics side and in quantum field theory and string theory — particularly its various geometry-related aspects — on the physics side. The soil is further enriched through the catalyzing effect of the mathematics-physics intertwining atmosphere in Yau's group and regular group meetings, though I had to admit that it's not very easy to catch up and keep my head above water in such an intense environment at the beginning and for a while I almost got drowned. It is only after the birth of this project at the end of 2006 that all this unusual, purely accidental luck given to me acquired its meaning. In retrospect, a project like this is very unlikely if not in such an intense and encompassing geometry-physics soil like the Boston area.

This work adds another special mark to the timeline of this project. Though clearly not in its final form (cf. Remark 3.2.4), for the first time since the beginning of this D-project the dynamics of D-branes is addressed along the line of the project in a most natural and geometric way. This brings the study of D-branes to the same starting point as that for the fundamental string: namely,

string theory	D-brane theory
<i>string world-sheet :</i> $2\text{-manifold } \Sigma$	<i>D-brane world-volume :</i> $Azumaya/matrix \text{ manifold}$ <i>with a fundamental module with a connection</i> (X^{Az}, E, ∇)
<i>string moving in space-time Y :</i> <i>differentiable map</i> $f : \Sigma \rightarrow Y$	<i>D-brane moving in space-time Y :</i> <i>differentiable map</i> $\varphi : (X^{Az}, E, \nabla) \rightarrow Y$
<i>Nambu-Goto action</i> $S_{\text{Nambu-Goto}}$ <i>for</i> f 's	<i>Dirac-Born-Infeld action</i> $S_{\text{Dirac-Born-Infeld}}$ <i>for</i> (φ, ∇) 's

While there are a long list of people I need to say thanks to (in particular, Shiraz Minwalla and Mihnea Popa, cf. D(6) Dedication), I would not be able to survive a decade's brewing to reach D(1), 2007, and then another seven years' brewing to reach D(11.1), 2014, without Ling-Miao. In a broad sense, this piece — indeed the whole D-project — is also her creation through her love. I thus dedicate this special mark of the project to her. Many challenging mathematical and physical issues of D-branes along the line remain ahead, some of them look beyond reach at the moment of writing, and I am still on the road for this journey that fills with unknowns.

0. Introduction and outline

In earlier works, [L-Y1] (arXiv:0709.1515 [math.AG], D(1)), [L-Y4] (arXiv:1406.0929 [math.DG], D(11.1)), [L-Y5] (arXiv:1412.0771 [hep-th], D(11.2)), and [L-Y6] (arXiv:1508.02347 [math.DG], D(11.3.1)), we have explained, and shown by feature stringy examples, why a D-brane in string theory, when treated as a fundamental dynamical object, can be described by a map φ from an Azumaya/matrix manifold X^{Az} , served as the D-brane world-volume, with a fundamental module with a connection (E, ∇) , served as the Chan-Paton bundle, to the target space-time Y . In this sequel, we construct a non-Abelian Dirac-Born-Infeld action functional $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$ for such pairs (φ, ∇) when the target space-time Y is equipped with a background (dilaton, metric, B)-field (Φ, g, B) from closed strings; (cf. Sec. 2 & Sec. 3). We next develop a technical tool needed to study variations of this action (cf. Sec. 4) and apply it to derive the first variation $\delta S_{DBI}^{(\Phi, g, B)} / \delta(\varphi, \nabla)$ of $S_{DBI}^{(\Phi, g, B)}$ with respect to (φ, ∇) (cf. Sec. 5). The equations of motion that govern the dynamics of D-branes then follow.

A complete action for a D-brane world-volume must include also the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C)}(\varphi, \nabla)$ that governs how the D-brane world-volume couples with the Ramond-Ramond fields C on Y . In the current notes, a version $S_{CS/WZ}^{(C, B)}(\varphi, \nabla)$ of non-Abelian Chern-Simons/Wess-Zumino action functional for (φ, ∇) that follows the same guide with which we construct $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$ is constructed for lower-dimensional D-branes (i.e. D(-1)-, D0-, D1-, D2-branes). Its first variation $\delta S_{CS/WZ}^{(C, B)}(\varphi, \nabla) / \delta(\varphi, \nabla)$ is derived and its contribution to the equations of motion for (φ, ∇) follows; (cf. Sec. 6). For D-branes of dimension ≥ 3 , an anomaly issue needs to be understood in the current context. The current notes lay down a foundation toward the dynamics of D-branes along the line of this D-project.

Some highlights of the history of how the Born-Infeld action and the Dirac-Born-Infeld action (cf. time-ordered: [Mie] (1912) of Gustav Mie, [Bo] (1934) of Max Born, [B-I] (1934) of Born and Leopold Infeld, [Di] (1962) of Paul Dirac) arise from open string theory and a list of issues one needs to resolve to convert such an action to that for coincident D-branes are given in Sec. 1. They serve as a guide for our discussion.

Convention. References for standard notations, terminology, operations and facts are

- (1) string theory: [B-B-S], [G-S-W], [Po3];
- (2) D-branes: [Joh], [Po3];
- (3) algebraic geometry: [Ha];
- (4) C^∞ -algebraic geometry: [Joy].

- For clarity, the *real line* as a real 1-dimensional manifold is denoted by \mathbb{R}^1 , while the *field of real numbers* is denoted by \mathbb{R} . Similarly, the *complex line* as a complex 1-dimensional manifold is denoted by \mathbb{C}^1 , while the *field of complex numbers* is denoted by \mathbb{C} .
- The inclusion ' $\mathbb{R} \subset \mathbb{C}$ ' is referred to the *field extension of \mathbb{R} to \mathbb{C}* by adding $\sqrt{-1}$, unless otherwise noted.
- The *real n -dimensional vector spaces* $\mathbb{R}^{\oplus n}$ vs. the *real n -manifold* \mathbb{R}^n ; similarly, the *complex r -dimensional vector space* $\mathbb{C}^{\oplus r}$ vs. the *complex r -fold* \mathbb{C}^r .
- All manifolds are paracompact, Hausdorff, and admitting a (locally finite) partition of unity. We adopt the *index convention for tensors* from differential geometry. In particular, the tuple coordinate functions on an n -manifold is denoted by, for example, (y^1, \dots, y^n) . However, no up-low index summation convention is used.

- For the current notes, ‘*differentiable*’, ‘*smooth*’, and C^∞ are taken as synonyms.
- For a smooth manifold X , $C^\infty(X) :=$ the ring of smooth functions on X .
For a vector bundle E over X , $C^\infty(E) :=$ the $C^\infty(X)$ -module of smooth sections of E .
- *wedge product* convention: For $\alpha \in C^\infty(\bigwedge^p X)$, $\beta \in C^\infty(\bigwedge^q M)$,

$$\begin{aligned}
& (\alpha \wedge \beta)(v_1, \dots, v_{p+q}) \\
& := \sum_{\substack{\sigma \in \text{Sym}_{p+q}; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p), \\ \sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)}} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \beta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}).
\end{aligned}$$

For example, $dx^1 \wedge dx^2 \wedge \dots \wedge dx^l := \sum_{\sigma \in \text{Sym}_l} (-1)^\sigma dx^{\sigma(1)} \otimes dx^{\sigma(2)} \otimes \dots \otimes dx^{\sigma(l)}$.

- *algebra* A_φ , sheaf of algebras \mathcal{A}_φ vs. *connection* 1-form A_μ .
- *degree* d vs. exterior *differential* d ; *diagonal* matrix D vs. covariant *derivative* D .
- *matrix* m vs. manifold of dimension m .
- the Regge slope α' vs. dummy labelling index α .
- *section* s of a fiber bundle vs. dummy labelling index s vs. coordinate s .
- *Chan-Paton bundle* E vs. the combined 2-tensor $g+B =: \sum_{i,j} E_{ij} dy^i \otimes dy^j$ from the metric tensor and the B -field.
- *ring* R vs. k -th *remainder* $R[k]$ vs. *Riemann* curvature tensor R_{ijkl} vs. index $(\cdot)^R$ for *right* factor or component.
- $\text{Spec } R$ ($:= \{\text{prime ideals of } R\}$) of a commutative Noetherian ring R in algebraic geometry vs. $\text{Spec } R$ of a C^k -ring R ($:= \text{Spec }^\mathbb{R} R := \{C^k\text{-ring homomorphisms } R \rightarrow \mathbb{R}\}$).
- *morphism* between schemes in algebraic geometry vs. C^k -*map* between C^k -manifolds or C^k -schemes in differential topology and geometry or C^k -algebraic geometry.
- The ‘*support*’ $\text{Supp}(\mathcal{F})$ of a quasi-coherent sheaf \mathcal{F} on a scheme Y in algebraic geometry or on a C^k -scheme in C^k -algebraic geometry means the *scheme-theoretical support* of \mathcal{F} unless otherwise noted; \mathcal{I}_Z denotes the *ideal sheaf* of a (resp. C^k -)subscheme of Z of a (resp. C^k -)scheme Y ; $l(\mathcal{F})$ denotes the *length* of a coherent sheaf \mathcal{F} of dimension 0.
- *coordinate-function index*, e.g. (y^1, \dots, y^n) for a real manifold vs. the *exponent of a power*, e.g. $a_0 y^r + a_1 y^{r-1} + \dots + a_{r-1} y + a_r \in \mathbb{R}[y]$.
- The current Notes D(13.1) continues the study in

[L-Y4] *D-branes and Azumaya/matrix noncommutative differential geometry, I: D-branes as fundamental objects in string theory and differentiable maps from Azumaya/matrix manifolds with a fundamental module to real manifolds*, arXiv:1406.0929 [math.DG]. (D(11.1))

[L-Y6] *Further studies on the notion of differentiable maps from Azumaya/matrix manifolds, I. The smooth case*, arXiv:1508.02347 [math.DG]. (D(11.3.1))

Notations and conventions follow these earlier works when applicable.

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1 Coincident D-branes and issues on non-Abelian Dirac-Born-Infeld action

Open-strings, background gauge fields, and the Born-Infeld action: The pre-D-brane era

Consider an open string moving in a space-time Y with a background dilaton field Φ , a background metric g , and a background B -field B , from the closed-string sector, and a $U(1)$ gauge field A , from the open-string sector. Then, similar to the governing equations for (Φ, g, B) from the conformal-anomaly-free conditions for the $2d$ field theory on the closed-string world-sheet, the dynamics of A is governed by the conformal-anomaly-free conditions on $2d$ field theory with boundary on the open-string world-sheet. It turns out that, at least for an appropriate lowest order approximation, this system of differential equations can be derived from the variation of the Lagrangian density¹

$$\mathcal{S}_{BI}^{(\Phi, g, B)}(\nabla) = -T e^{-\Phi} \sqrt{-\text{Det}(g + B + 2\pi\alpha' F_{\nabla})}.$$

Here, T is a physical constant, F_{∇} is the curvature of ∇ , and the determinant Det is for the 2-tensor $g + B + 2\pi\alpha' F_{\nabla}$ on Y . (The fields (Φ, g, B) are governed by another Lagrangian density $\mathcal{S}(\Phi, g, B)$, we will completely omit as they are irrelevant to our discussion.) Furthermore, through the coupling to the Chan-Paton index, the above (Abelian) Born-Infeld action is expected to be generalizable to a non-Abelian Born-Infeld action

$$\mathcal{S}_{BI}^{(\Phi, g, B)}(\nabla) = -T \cdot \text{STr} \left(e^{-\Phi} \sqrt{-\text{Det}(g + B + 2\pi\alpha' F_{\nabla})} \right).$$

Here, the STr is the ‘*symmetrized trace*’, with

- the *trace-part* Tr in STr serving as a perturbative method to understand $\sqrt{-\text{Det}(\cdots)}$ in terms of an expansion at $g + B$ through powers of α' via the Taylor series of the analytic formula

$$(\text{Det}(1 + x))^{\beta} = \exp(\beta \text{Tr}(\text{Log}(1 + x)))$$

after some natural manipulation of the expression of $\mathcal{S}_{BI}^{(\Phi, g, B)}(\nabla)$; this takes care of the meaning of $\sqrt{-\text{Det}(\cdots)}$ for a Lie-algebra valued 2-tensor (\cdots) that appears;

- the *symmetrized-part* S in STr taking in addition into account the fact that the Lie algebra involved now is non-Abelian:

$$\text{STr}(m_1 \cdots m_l) := \frac{1}{l!} \sum_{\sigma \in \text{Sym}_l} \text{Tr}(m_{\sigma(1)} \cdots m_{\sigma(l)})$$

for m_1, \cdots, m_l in the Lie algebra in matrix form; not only that this is a very natural thing to do to deal with the noncommutativity issue here, one anticipates that this symmetrization is also required to fit the perturbative result better with the open-string consideration.

Readers are referred to, for example,² (time-ordered) [F-T] (1985) of *Efim Fradkin* and *Arkady Tseytlin*, [D-O] (1986) of *Harald Dorn* and *Hans-Jorg Otto*, [A-C-N-Y] (1987) of *Ahmed Abouelsaood*, *Curtis Callan, Jr.*, *Chiara Nappi*, and *Scott Yost* (preprint: 1986), [A-N] (1990) of *Philip*

¹Here, some mild natural, well-accepted updates and change of notations are made to the quoted original works in order to fit in better the current notes. Our apology to the original authors.

²(From C.H.L.) These are some works that particularly influenced my thought during my brewing years 2007-2015 on this topic. There is no intention at all to make a complete survey on this topic here. Readers should consult references therein and key-word search for a more comprehensive understanding of this part of the history of the development. Similarly, for the literature on its upgrade: the Dirac-Born-Infeld action.

Argyres and *Nappi* (preprint: 1989) in the second half of 1980s for more details, further references, and issues that remain.

In the language of nowadays, all these studies are in the special case where the space-time is filled with a D-brane-world-volume so that the end-point of open strings can move freely anywhere in the space-time. (In the notation of the next theme, this means that $X = Y$, with the background fields (Φ, g, B) living on the space-time Y and ∇ living on the Chan-Paton bundle E over the D-brane world-volume X .) Which we now turn to.

D-branes, Dirac-Born-Infeld action, and its non-Abelian generalization

A *D-brane*, in full name: *Dirichlet brane*, in string theory is by definition (i.e. by the very word ‘Dirichlet’) a boundary condition for the end-points of open strings. From the viewpoint of the field theory on the open-string world-sheet aspect, it is a boundary state in the 2-dimensional conformal field theory with boundary. From the viewpoint of open-string target-space(-time) Y , its world-volume is a cycle or a union of submanifolds X in Y with a Chan-Paton bundle with a $U(1)$ -connection (E, ∇) , supported on X , that carries the Chan-Paton index for the end-points of oriented open strings, cf. FIGURE 1-1.

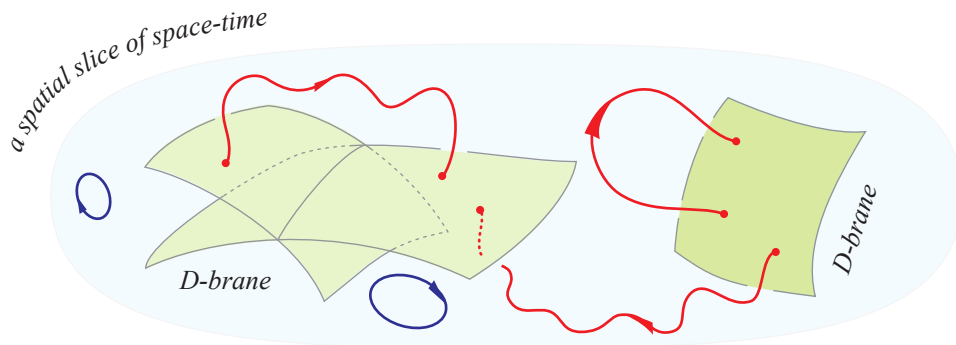


FIGURE 1-1. D-branes as boundary conditions for open strings in space-time. This gives rise to interactions of D-brane world-volumes with open strings. Properties of D-branes, including the quantum field theory on their world-volume and deformations of such, are governed by open strings via this interaction. Both oriented open (resp. closed) strings and a D-brane configuration are shown.

In the region of Wilson’s theory-space of string theory where the D-brane tension is small, D-branes stand in an equal footing with strings as fundamental objects. In this region, they are soft and can move around and vibrate, just like a fundamental string can, in the space-time Y . Thus, a D-brane world-volume in this case is better described as a map $f : X \rightarrow Y$. Such non-solitonic aspect was already taken in the original works, [P-C] (1989) of *Joseph Polchinski* and *Yunhai Cai* and [D-L-P] (1989) of *Jin Dai*, *Robert Leigh*, and *Joseph Polchinski*, that introduced the notion of D-branes to string theory. With the earlier works that relate the Born-Infeld action $S_{BI}^{(\Phi, g, B)}(\nabla)$ to the dynamics of ∇ that couple to the open-string in the special space-time-filling-D-brane case where $X = Y$, it is immediately realized by *Leigh* [Le] (1989) that the action for such a simple D-brane (f, ∇) is given the Dirac-Born-Infeld action density

$$S_{DBI}^{(\Phi, g, B)}(f, \nabla) = -T_{m-1} e^{-\Phi} \sqrt{-\text{Det}(f^*(g + B) + 2\pi\alpha' F_{\nabla})}$$

so that the resulting system of equations of motion for a simple D-brane (f, ∇) coincides with the system of conformal-anomaly-free constraint equations on (f, ∇) from the aspect of $2d$

boundary conformal field theory on the open-string world-sheet. Here, $m = \dim X$, $m - 1$ is the dimension of the D-brane and T_{m-1} is the $D(m - 1)$ -brane tension. While this is a very natural generalization of the earlier work, it is worth emphasizing that

- *This is an action for the pair (f, ∇) . It governs not only how the gauge field ∇ lives on the Chan-Paton bundle E on the D-brane world-volume X but also how the D-brane world-volume sits in the space-time Y , i.e. the map $f : X \rightarrow Y$.*

Fast forward now to year 1995, *Polchinski* [Po1] (1995) realized that D-branes serve not only as general boundary conditions for open strings but, in a superstring theory, can also couple to the Ramond-Ramond fields created by closed superstrings, and, hence, serve as the source for such fields. This shifted the focus of string theory from strings to the various higher-dimensional extended objects: branes.

Something novel and mysterious at the first sight happens when a collection of D-branes in space-time coincide: (cf. [Wi] (1995) of *Edward Witten* and [Po3] (1996) and [Po4] (1998) of *Polchinski*)

- **[enhancement of scalar field on D-brane world-volume]** *When a collection of D-branes in space-time coincide, the open-string-induced massless spectrum on the world-volume of the D-brane is enhanced. In particular, the gauge field is enhanced to one with a larger gauge group and the scalar field that describes the deformations of the brane in space-time is enhanced to one that is matrix-valued.*

Cf. FIGURE 1-2.

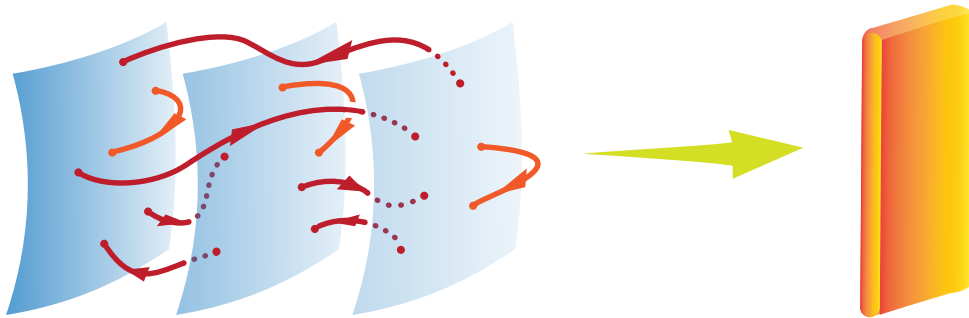


FIGURE 1-2. When a collection of D-branes coincide, the massless spectrum on the common world-volume are enhanced. Not only that the gauge field becomes non-Abelian, the scalar field is also enhanced and becomes non-Abelian.

This leads to the following key guiding questions:

- Q1. [D-brane]** *What is a D-brane as a fundamental object (as opposed to a solitonic object) in string theory?*
- Q2. [dynamics]** *What rules govern its dynamics?*

In other words, what is the intrinsic definition of D-branes so that by itself it can produce the properties of D-branes that are consistent with, governed by, or originally produced by open strings as well?

Leaving the first question — which is even more fundamental of the two — tentatively aside, with this new motivation further attempts to generalize the Born-Infeld or Dirac-Born-Infeld in the Abelian case (i.e. for a simple D-brane) to a non-Abelian case (i.e. for coincident D-branes) followed suit immediately: for example, [Dor1] (1996) and [Dor2] (1997) of *Dorn*, [Ts1] (1997)

and [Ts2] (1999) of *Tseytlin*, [B-dR-S1] (2000) and [B-dR-S2] (2000) of *Eric Bergshoeff*, *Mees de Roo*, and *Alexander Sevrin*, [Schw] (2001) of *John Schwarz*, [My] (2001) of *Robert Myers*, and the thesis [Sé] (2005) of *Emmanuel Sérié*. Readers are referred to these works for details, further references, and issues that remain.

The construction of a non-Abelian Dirac-Born-Infeld action and its consequences

Back to the two guiding questions, for the first that concerns the intrinsic nature of D-branes, in the work [H-W: Sec. 5] (1996) of *Pei-Ming Ho* and *Yong-Shi Wu* they realized that (assuming that the Chan-Paton bundle is a trivialized trivial complex vector bundle of rank r) a D-brane world-volume X carries a full *matrix-ring structure* $M_{r \times r}(L^2(X))$, where $L^2(X)$ is the Hilbert space of square-integrable functions on X . A decade afterwards, in late 2006 their important observation was re-picked up by the first author of the current notes when he re-thought of the lecture [Po2] of *Polchinski* from the viewpoint of *Grothendieck's* Modern Algebraic Geometry. Such input from algebraic geometry (cf. [Ha] (1977) and [Joy] (2010)) gave rise to a proto-typical definition of D-branes ([L-Y1] (D(1), 2007; [L-Y4] (D(11.1), 2014; [L-Y6] (D(11.3.1), 2015)):

Ansatz/Definition 1.1. [D-brane: prototypical] Let X be the world-volume of a D-brane (i.e. a C^∞ -manifold), E be the Chan-Paton bundle on X (i.e. a complex vector bundle of rank r) on X . Then a *D-brane moving in a space-time* Y is modelled on a ‘map’

$$\varphi : (X^{A_z}, E; \nabla) \longrightarrow Y ,$$

where

- $(X^{A_z}, E) := (X, C^\infty(\text{End}_{\mathbb{C}}(E)), E)$ is an *Azumaya/matrix manifold with a fundamenatl module* whose underlying topology is identical to the manifold X but whose function-ring is given by the *endomorphism-algebra* $C^\infty(\text{End}_{\mathbb{C}}(E))$ over \mathbb{C} ,
- ∇ is a connection on E .

Here, the notion of a ‘map’ is *defined contravariantly by a ring-homomorphism*

$$\varphi^\# : C^\infty(Y) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$$

over the canonical inclusion $\mathbb{R} \subset \mathbb{C}$. Cf. FIGURE 1-3.

Readers are referred to Sec. 2.1 of the current notes for a terse review of the part/notation we need and to [L-Y1] (D(1)), [L-L-L-Y] (D(2)), [L-Y3] (D(6)), [L-Y4] (D(11.1)), [L-Y5] (D(11.2)), [L-Y6] (D(11.3.1)) for further details, examples, and the justification by comparing to various D-brany phenomena in string theory that this definition is workable and does capture some major features of D-branes.

Once having a proto-typical answer to Question 1, we now turn to Question 2. Taking the lesson from string-theorists that the dynamics of D-branes should be governed (at least at the lowest level) by a generalization of the Dirac-Born-Infeld action in the abelian case (i.e. the case where E is a complex line bundle over X), let us write down first a formal but natural expression for the Dirac-Born-Infeld action on the pairs (φ, ∇) :

$$S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla) \stackrel{\text{formally}}{=} -T_{m-1} \int_X \text{Tr} \left(e^{-\varphi^*(\Phi)} \sqrt{-\text{Det}_X(\varphi^*(g + B) + 2\pi\alpha' F_\nabla)} \right) .$$

Here, $m = \dim X$ and T_{m-1} is the tension fo the $D(m-1)$ -brane. We now list all the issues that need to be resolved to make sense, or interpret correctly, of this formal expression:

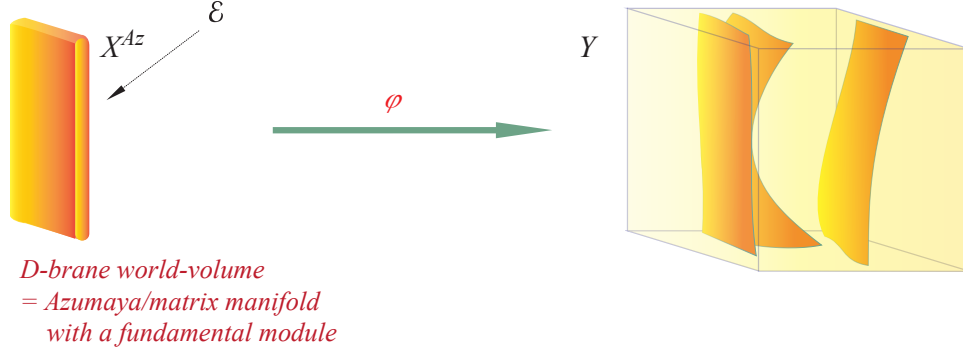


FIGURE 1-3. As a dynamical object in string theory, D-brane moving in a space-time Y can be described as a map φ from an Azumaya/matrix manifold X^{4z} with a fundamental module E with a connection ∇ to Y .

(1) [map φ]

We have settled down the notion of φ purely algebro-geometrically, without having anything to do with the connection ∇ . Once the connection is brought into play, *is there a constraint on the pair (φ, ∇) that comes from string-theoretical consideration?* Furthermore, *when E is Hermitian and ∇ is unitary, is there a class of maps φ that stand out from others due to such additional structure on (E, ∇) ?*

(2) [push-pull of tensor under φ]

The notion of *push-pulls under a differentiable map φ* from an Azumaya/matrix manifold with a fundamental module to a real manifold; cf. $\varphi^*(g + B)$.

(3) [determinant of 2-tensor on X]

The notion of *determinant $\text{Det}_X(\cdots)$* in the current context once Issues (1) and (2) are resolved.

(4) [square root of matrix section]

Can we take a *square root of a matrix*? When the answer is Yes, is there a unique square root? If not, which one to choose? Extension of this to *matrix sections*?

(5) [dilaton-field factor]

How does *the factor $e^{-\varphi^\sharp(\Phi)}$* influence the interpretation of the formal expression?

(6) [real-valuedness]

Is the expression *real*?

(7) [consistency with open string theory]

How does it fit with open string theory?

In this work, we will answer to and resolve Issues (1) - (6) above in a way that is physically meaningful and mathematically natural and construct a Dirac-Born-Infeld action S_{DBI} for (φ, ∇) (Sec. 2 and Sec. 3). We then develop a necessary tool (Sec. 4) to carry out the first variation formula of S_{DBI} with respect to (φ, ∇) (Sec. 5). In view of Polchinski's realization ([Po1]) that D-brane world-volume serves as the source for Ramond-Ramond fields in superstring theory, the Chern-Simons/Wess-Zumino action $S_{CS/WZ}$ for D-branes is also an indispensable part to understand the dynamics of D-branes. With the same essence as for the construction of $S_{DBI}(\varphi, \nabla)$, we construct the Chern-Simons/Wess-Zumino action $S_{CS/WZ}(\varphi, \nabla)$

in Sec. 6 for lower-dimensional D-branes, in which cases anomaly issues do not occur, derive their first variation formula and, hence, obtain their contribution to the equations of motions for D-branes.

Remark 1.2. [effect of B-field to fundamental module E] The presence of a B -field on the space-time Y can have a non-trivial twisting effect to the Chan-Paton bundle E on the D-brane world-volume X , rendering it no longer an honest vector bundle but, rather, a ‘twisted vector bundle’ ([Wi]). For better focus, we omit this effect in the current notes. Reader are referred to [Wi] and, e.g., [Kap] for more details and references, and to [L-Y2] (D(5)) for details on how this effect is taken into account in our setting. This twisting effect can always be added back to our presentation whenever in need.

2 Differentiable maps from an Azumaya/matrix manifold with a fundamental module with a connection

Recall from Sec. 1 the first issue we need to understand before we can construct the Dirac-Born-Infeld action for D-branes in our setting:

(1) *[map φ]*

We have settled down the notion of φ purely algebro-geometrically, without having anything to do with the connection ∇ . Once the connection is brought into play, *is there a constraint on the pair (φ, ∇) that comes from string-theoretical consideration?* Furthermore, *when E is Hermitian and ∇ is unitary, is there a class of maps φ that stand out from others due to such additional structure on (E, ∇) ?*

In this section, we first review very tersely the part of [L-Y4] (D(11.1)) and [L-Y6] (D(11.3.1)) that is needed for the current notes (Sec. 2.1), then address a compatibility issue from the open-string aspect between our notion of maps φ from a matrix manifold to a space-time and the connection ∇ on the fundamental module associated to that matrix manifold (Sec. 2.2), and finally bring out the additional notion of *self-adjoint map* in our context (Sec. 2.3). This notion is what we interpret the (seemingly Lie-algebra-valued) scalar field on the common world-volume of coincident D-branes as in [Po2], [Po3], and [Wi] when the connection on the Chan-Paton bundle is unitary.

2.1 Differentiable maps from an Azumaya/matrix manifold with a fundamental module

Definition 2.1.1. [map from Azumaya/matrix manifold] Let X be a (real, smooth) manifold, E be a complex vector bundle of rank r over X , and $(X^{\text{Az}}, E) := (X, C^\infty(\text{End}_{\mathbb{C}}(E)), E)$ be the associated Azumaya/matrix manifold with a fundamental module. A *map* (synonymously, *differentiable map*, *smooth map*)

$$\varphi : (X^{\text{Az}}, E) \longrightarrow Y$$

from (X^{Az}, E) to a (real, smooth) manifold Y is defined by a ring-homomorphism

$$\varphi^\# : C^\infty(Y) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E)).$$

Definition 2.1.2. [push-forward $\varphi_*\mathcal{E}$] Let \mathcal{E} be the sheaf of (smooth) sections of E ; it is canonically identical to the sheaf on X from localizations of $C^\infty(E)$. The ring-homomorphism $\varphi^\sharp : C^\infty \rightarrow C^\infty(\text{End}_{\mathbb{C}})$ renders $C^\infty(E)$ a $C^\infty(Y)$ -module. This defines a sheaf on Y , denoted by $\varphi_*\mathcal{E}$, and is called the *push-forward* of \mathcal{E} under φ . It is an $\mathcal{O}_Y^{\mathbb{C}}$ -module.

Proposition 2.1.3. [basic properties of φ^\sharp] (1) [realness nature of φ^\sharp] For any $f \in C^\infty(Y)$ and $x \in X$ (an \mathbb{R} -point), the eigenvalues of $\varphi^\sharp(f)|_x \in \text{End}_{\mathbb{C}}(E|_x) \simeq M_{r \times r}(\mathbb{C})$ are all real. (Cf. [L-Y4: Sec. 3] (D(11.1)).)

(2) [canonical lifting to $C^\infty(X \times Y)$] The ring-homomorphism

$$C^\infty(\text{End}_{\mathbb{C}}(E)) \xleftarrow{\varphi^\sharp} C^\infty(Y)$$

extends canonically to a commutative diagram of ring-homomorphisms (over \mathbb{R} or $\mathbb{R} \subset \mathbb{C}$, whichever is applicable)

$$\begin{array}{ccc} C^\infty(\text{End}_{\mathbb{C}}(E)) & \xleftarrow{\varphi^\sharp} & C^\infty(Y) \\ \uparrow & \nwarrow \tilde{\varphi}^\sharp & \downarrow pr_Y^\sharp \\ C^\infty(X) & \xrightarrow{pr_X^\sharp} & C^\infty(X \times Y), \end{array}$$

where $pr_X : X \times Y \rightarrow X$ and $pr_Y : X \times Y \rightarrow Y$ are the projection maps, and $C^\infty(X) \hookrightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ follows from the inclusion of the center $C^\infty(X)^{\mathbb{C}}$ of $C^\infty(\text{End}_{\mathbb{C}}(E))$. (Cf. [L-Y6: Theorem 3.1.1] (D(11.3.1)).)

Definition 2.1.4. [graph of φ] The above diagram of ring-homomorphisms defines a commutative diagram of maps

$$\begin{array}{ccc} (X^{Az}, E) & \xrightarrow{\varphi_T} & Y \\ \downarrow & \searrow \tilde{\varphi} & \uparrow pr_Y \\ X & \xleftarrow{pr_X} & X \times Y. \end{array}$$

The push-forward $\tilde{\varphi}_*\mathcal{E} =: \tilde{\mathcal{E}}_\varphi$ of \mathcal{E} under $\tilde{\varphi}$ is called the *graph* of φ . It is an $\mathcal{O}_{X \times Y}^{\mathbb{C}}$ -module. Its C^∞ -scheme-theoretical support is denoted by $\text{Supp}(\tilde{\mathcal{E}}_\varphi)$.

Definition 2.1.5. [surrogate of X^{Az} specified by φ] The image

$$A_\varphi := \text{Im } \tilde{\varphi}^\sharp := \tilde{\varphi}(C^\infty(X \times Y)) = C^\infty(X) \langle \text{Im } \varphi^\sharp \rangle \subset C^\infty(\text{End}_{\mathbb{C}}(E))$$

of $\tilde{\varphi} : C^\infty(X \times Y) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ is a commutative $C^\infty(X)$ -subalgebra of $C^\infty(\text{End}_{\mathbb{C}}(E))$ that is locally algebraically finite over $C^\infty(X)$. It defines a C^∞ -scheme

$$X_\varphi := \text{Spec}^{\mathbb{R}}(A_\varphi),$$

which is called the *surrogate* of X^{Az} specified by φ . X_φ is finite over X and, by construction, it admits a canonical embedding $\tilde{f}_\varphi : X_\varphi \rightarrow X \times Y$ into $X \times Y$ as a C^∞ -subscheme. The image is identical to $\text{Supp}(\tilde{\mathcal{E}}_\varphi)$. Cf. FIGURE 2-1-1.

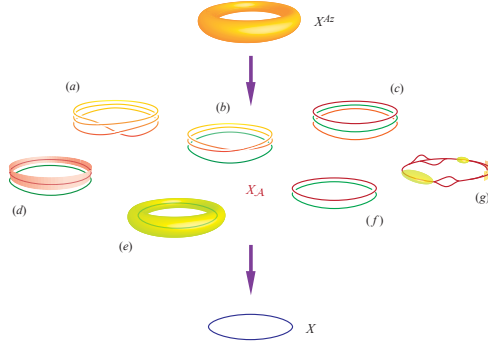


FIGURE 2-1-1. The C^∞ -scheme $X_A := \text{Spec}^\mathbb{R} A$ associated to a commutative $C^\infty(X)$ -subalgebra $C^\infty(X) \subset A \subset C^\infty(\text{End}_\mathbb{C}(E))$ can be thought of as interpolating between the commutative X and the noncommutative X^{Az} by the built-in dominant maps

$$X^{Az} \twoheadrightarrow X_A \twoheadrightarrow X.$$

The abundance of such objects under X^{Az} indicates a very rich geometric structure the Azumaya/matrix manifold X^{Az} contains. In the Figure, seven surrogates (a) – (g) of an Azumaya/matrix string $S^{1,Az}$ are indicated. They include short-string sets: (c) and (f), a long string (a), a fuzzy string (e), and various mixtures: (b), (d), (g). In particular, given a map $\varphi : (X^{Az}, E) \rightarrow Y$, the surrogate of X^{Az} specified by φ can be used to help capture φ itself and serve as a φ -specified medium between X and Y .

One can summarize all the objects introduced into the following two diagrams that refine the contravariant pair of diagrams in Proposition 2.1.3 (2) and Definition 2.1.4 respectively:

$$\begin{array}{c}
 C^\infty(E) \\
 \swarrow \quad \searrow \\
 C^\infty(\text{End}_\mathbb{C}(E)) \quad \xleftarrow{\varphi^\#} \quad C^\infty(Y) \\
 \uparrow \quad \quad \quad \downarrow \pi_\varphi^\# \\
 A_\varphi \quad \xleftarrow{f_\varphi^\#} \quad C^\infty(X \times Y) \\
 \uparrow \pi_\varphi^\# \quad \quad \quad \downarrow pr_Y^\# \\
 C^\infty(X) \quad \xrightarrow{pr_X^\#} \quad C^\infty(X \times Y)
 \end{array}$$

and

$$\begin{array}{c}
 \mathcal{E} \\
 \swarrow \quad \searrow \\
 X^{Az} \quad \xrightarrow{\varphi} \quad Y \\
 \downarrow \sigma_\varphi \quad \quad \quad \downarrow pr_Y \\
 X_\varphi \quad \xrightarrow{f_\varphi} \quad Y \\
 \downarrow \pi_\varphi \quad \quad \quad \downarrow pr_X \\
 X \quad \xleftarrow{pr_X} \quad X \times Y
 \end{array} ,$$

with the built-in isomorphisms

$$\mathcal{E} \simeq \pi_{\varphi*}(\mathcal{O}_{X_{\varphi}} \mathcal{E}) \simeq (\pi_{\varphi} \circ \sigma_{\varphi})_*(\mathcal{O}_X^{\text{Az}} \mathcal{E}) \quad \text{and} \quad \mathcal{O}_{X_{\varphi}} \mathcal{E} \simeq \sigma_{\varphi*}(\mathcal{O}_X^{\text{Az}} \mathcal{E}).$$

At this point, readers may feel that such a notion of maps is too abstract to perceive. Recall then that in the ordinary differential topology or geometry, a map f from a manifold X to \mathbb{R}^n , with coordinates (y^1, \dots, y^n) , is determined by specifying its projection to each coordinate, i.e. $f = (f^1, \dots, f^n) : X \rightarrow \mathbb{R}^n$. Each f^i is now in $C^\infty(X)$. Thus, in terms of function-rings, this means that f is determined by the n -tuple $(f^\sharp(y^1), \dots, f^\sharp(y^n))$, which is exactly the n -tuple (f^1, \dots, f^n) above. The following proposition from [L-Y6] (D(11.3.1)) says that a very similar statement holds for our notion of maps from Azumaya/matrix manifolds:

Proposition 2.1.6. [map from Azumaya/matrix manifold to \mathbb{R}^n] ([L-Y6: Theorem 3.2.1] (D(11.3.1))). *Let X be a smooth manifold and E be a complex smooth vector bundle of rank r on X . Let (y^1, \dots, y^n) be a global coordinate system on \mathbb{R}^n , as a smooth manifold, and*

$$\eta : y^i \longmapsto m_i \in C^\infty(\text{End}_{\mathbb{C}}(E)), \quad i = 1, \dots, n,$$

be an assignment such that

- (1) $m_i m_j = m_j m_i$, for all i, j ;
- (2) for every $p \in X$, the eigenvalues of the restriction $m_i(p) \in \text{End}_{\mathbb{C}}(E|_p) \simeq M_{r \times r}(\mathbb{C})$ are all real.

Then, η extends to a unique ring-homomorphism

$$\varphi_\eta^\sharp : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$$

over $\mathbb{R} \subset \mathbb{C}$ and, hence, defines a map $\varphi_\eta : (X^{\text{Az}}, E) \rightarrow \mathbb{R}^n$.

This is a consequence of the Malgrange Division Theorem ([Mal]; also [Brö], [Mat1], [Mat2], [Ni]). Note that Conditions (1) and (2) in Proposition 2.1.6 are necessary conditions for η to be extendable to a full ring-homomorphism. The proposition says that they are also sufficient and the extension is unique. Due to its importance as a technical tool for our study later, we will highlight its proof in Sec. 4.1 in the form we need and then generalize it to a similar statement for derivatives $\partial^\alpha \varphi^\sharp$ of φ^\sharp to all orders $|\alpha|$ in Sec. 4.2. It is through this proposition that we can almost visualize φ^\sharp as we would for f .

Example 2.1.7. [D0-brane on \mathbb{R}^2 , deformation, Higgsing/un-Higgsing] D0-branes (or D(-1)-brane world-points) on \mathbb{R}^2 can be described by maps from Azumaya/matrix points $\varphi : (p^{\text{Az}}, \mathbb{C}^{\oplus r}) \rightarrow \mathbb{R}^2$, defined by ring-homomorphisms

$$\varphi^\sharp : C^\infty(\mathbb{R}^2) \longrightarrow M_{r \times r}(\mathbb{C}).$$

The latter is determined by the value $(m^1, m^2) := (\varphi^\sharp(y^1), \varphi^\sharp(y^2)) \in M_{r \times r}(\mathbb{C}) \times M_{r \times r}(\mathbb{C})$ of φ^\sharp on the coordinates (y^1, y^2) of \mathbb{R}^2 . Any pair (m^1, m^2) of $r \times r$ matrices (with entries in \mathbb{C}) that commute and with each matrix having only real eigenvalues defines a φ . Deformations of φ may create various Higgsing/un-Higgsing phenomena of D0-branes on \mathbb{R}^2 . Cf. FIGURE 2-1-2.

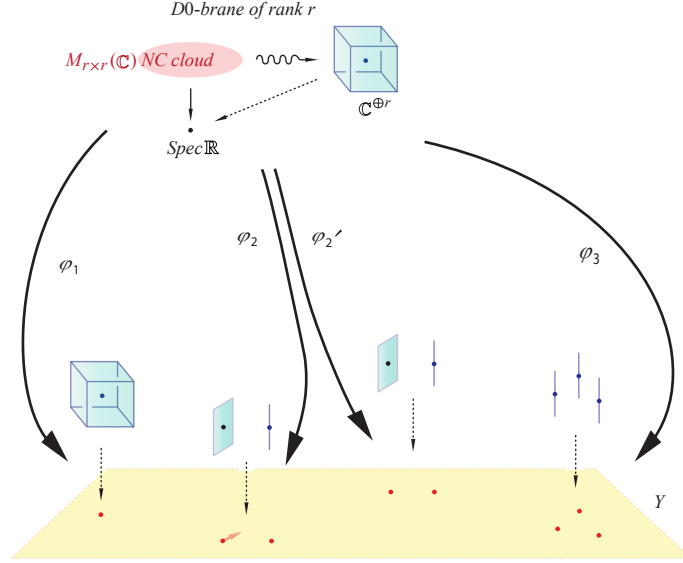


FIGURE 2-1-2. Four examples of maps $\varphi : (p^{Az}, \mathbb{C}^{\oplus r}) \rightarrow \mathbb{R}^2$ from an Azumaya/matrix point with a fundamental module to \mathbb{R}^2 are illustrated. The nilpotency of the image scheme $Im \varphi$ in \mathbb{R}^2 is bounded by r . In the figure, the push-forward $\varphi_*(\mathbb{C}^{\oplus r})$ of the fundamental module in each example is also indicated.

Remark 2.1.8. [*Dp-brane from smearing D0-branes*] Functionally, a Dp -brane can be thought of as from smearing a jam of D0-branes along a p -cycle X . A map $\varphi : X^{Az} \rightarrow Y$ may be thought of as an X -family of maps $\varphi_x : p^{Az} \rightarrow Y$ from an Azumaya/matrix point p^{Az} to Y . This gives another way to visualize φ , in addition to the picture by the surrogate X_φ of X^{Az} specified by φ . Cf. FIGURE 2-1-3 and Example 2.1.7.

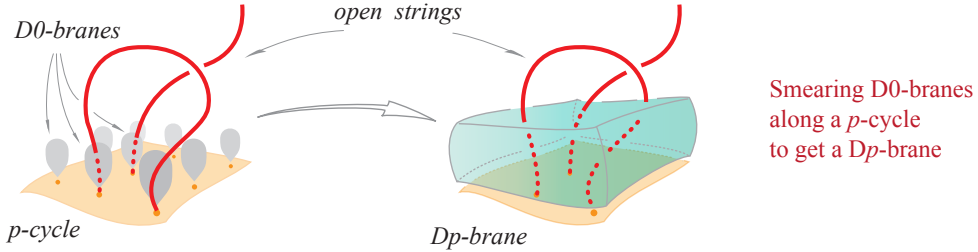


FIGURE 2-1-3. A Dp -brane from smearing a jam of D0-branes along a p -cycle.

2.2 Compatibility between the map φ and the connection ∇ from the open-string aspect

Recall ([DV-M: Proposition 3]; see ibidem and [L-Y4: Sec. 4] for more references and discussions) that a connection ∇ on a complex vector bundle E over X induces canonically a connection D on the endomorphism bundle $End_{\mathbb{C}}(E) (\simeq E \otimes_{\mathbb{C}} E^{\vee})$ canonically, with E^{\vee} the dual vector bundle of E). Let $\pi_X^{\sharp} : C^{\infty}(X) \hookrightarrow C^{\infty}(End_{\mathbb{C}}(E))$ be the inclusion that follows from the inclusion of

center $C^\infty(X)^\mathbb{C} \subset C^\infty(\text{End}_\mathbb{C}(E))$, \mathcal{T}_*X , $\mathcal{T}_*X^{A_z}$ be the tangent sheaf (i.e. sheaf of derivations) of X and X^{A_z} respectively, and $\mathcal{I}nn(\mathcal{O}_X^{A_z})$ be the sheaf of inner derivations of the structure sheaf $\mathcal{O}_X^{A_z}$. Then there is a natural exact sequence of $\mathcal{O}_X^\mathbb{C}$ -modules

$$0 \longrightarrow \mathcal{I}nn(\mathcal{O}_X^{A_z}) \longrightarrow \mathcal{T}_*X^{A_z} \longrightarrow \mathcal{T}_*X^\mathbb{C} \longrightarrow 0.$$

Furthermore, since

$$D_\bullet(m_1 m_2) = (D_\bullet m_1) m_2 + m_1 D_\bullet m_2$$

for all $m_1, m_2 \in C^\infty(\text{End}_\mathbb{C}(E))$, the connection $\nabla : \mathcal{E} \rightarrow \mathcal{T}_*X \otimes_{\mathcal{O}_X^\mathbb{C}} \mathcal{E}$ on \mathcal{E} induces an embedding

$$\iota^\nabla : \mathcal{T}_*X^\mathbb{C} \hookrightarrow \mathcal{T}_*X^{A_z}$$

as $\mathcal{O}_X^\mathbb{C}$ -modules, with $\xi \mapsto D_\xi$, that splits the above short exact sequence. D satisfies the property that

$$\pi_X^\# df(\bullet) = D_\bullet \pi_X^\#(f)$$

for all $f \in C^\infty(X)$. All these serve as an indication that, in a sense, X^{A_z} is flatly uniform over X differentially topologically. Which renders the differential calculus on X^{A_z} accessible with respect to that on X , despite being noncommutative.

In contrast, when given a map $\varphi : (X^{A_z}, E) \rightarrow Y$, the surrogate X_φ of X^{A_z} specified by φ , though a commutative C^∞ -scheme finite over X , may not be flat or uniform over X . And it has nothing to do with ∇ at all. Since the fundamental (left) $\mathcal{O}_{X^{A_z}}$ -module \mathcal{E} descends canonically to an \mathcal{O}_{X_φ} -module ${}_{\mathcal{O}_{X_\varphi}}\mathcal{E}$, one would like ∇ induces canonically a connection ${}^\varphi\nabla$ on ${}_{\mathcal{O}_{X_\varphi}}\mathcal{E}$ at least over an open-dense subset of X . When that happens, the curvature F_∇ of ∇ should behave also as a tensor, possibly with singularity, on X_φ .

These together motivate us the following definition:

Definition 2.2.1. [admissible pair (φ, ∇)] Let $\varphi : (X^{A_z}, E) \rightarrow Y$ be a differentiable map and ∇ be a connection on E . The pair (φ, ∇) is called *admissible* if the following two conditions are satisfied over an open-dense subset of X :

$$(1) \quad DA_\varphi \subset C^\infty(\Omega_X) \otimes_{C^\infty(X)} A_\varphi \quad \text{and} \quad (2) \quad F_\nabla \subset C^\infty(\Omega_X^2) \otimes_{C^\infty(X)} \text{Comm}(A_\varphi).$$

Here, $\text{Comm}(A_\varphi)$ is the commutant of A_φ in $C^\infty(\text{End}_\mathbb{C}(E))$. For convenience, we say also that φ is an *admissible map* from $(X^{A_z}, E; \nabla)$ to Y , or that $\varphi : (X^{A_z}, E) \rightarrow Y$ is a map that is *admissible* to ∇ on E , or that ∇ is a connection on E that is *admissible* to $\varphi : (X^{A_z}, E) \rightarrow Y$.

Further illuminations of Definition 2.2.1 are given in the following two remarks:

Remark 2.2.2. [on Admissibility Condition (1) : generic uniformity of X_φ over X] The Admissible Condition (1) says that

- The commutative $C^\infty(X)$ -subalgebra A_φ of $C^\infty(\text{End}_\mathbb{C}(E))$ is covariantly invariant under the induced connection D on $\text{End}_\mathbb{C}(E)$ over an open-dense subset U of X .

This defines an embedding

$$\text{Der}(C^\infty(U)) \subset \text{Der}(A_\varphi|_U)$$

and, hence, a connection on $X_\varphi|_U$ over U . In terms of the above inclusion,

- One can associate to a tensor of type $(0, d)$ on $X_\varphi|_U$ an $A_\varphi|_U$ -valued tensor of the same type $(0, d)$ on U , which is then canonically an $C^\infty(\text{End}_\mathbb{C}(E|_U))$ -valued tensor of type $(0, d)$ on U through the localization of the built-in embedding $A_\varphi \subset C^\infty(\text{End}_\mathbb{C}(E))$ to over U .

Remark 2.2.3. [on Admissibility Condition (2) : massless condition on ∇ with respect to open strings] When D -brane is treated as a fundamental dynamical object, its interaction with open strings is through its image in the space-time Y . For simplicity, assume that $pr_Y : X \times Y \rightarrow Y$ pushes $\tilde{\mathcal{E}}_\varphi$ to $\varphi_*(\mathcal{E})$ isomorphically and, hence, $X_\varphi \simeq \text{Supp}(\varphi_*(\mathcal{E}))$. Then, if ∇ is to be massless from the aspect of open strings moving in Y , ∇ must be descendable to a connection ${}_\varphi\nabla$ on $\mathcal{O}_{X_\varphi}\mathcal{E}$ over X_φ . When that happens, its curvature $F_{{}_\varphi\nabla}$ becomes a $(0, 2)$ -tensor on X_φ and, hence, takes values in the commutant $\text{Comm}(A_\varphi)$ of A_φ in $C^\infty(\text{End}_{\mathbb{C}}(E))$. When, in addition, the Admissible Condition (1) holds, to $F_{{}_\varphi\nabla}$ is associated a $\text{End}_{\mathbb{C}}(E)$ -valued 2-form on X , which is nothing but F_∇ . Thus, one has the Admissible Condition (2): $F_\nabla \subset C^\infty(\Omega_X^2) \otimes_{C^\infty(X)} \text{Comm}(A_\varphi)$. This reasoning indicates that

- *Admissible Condition (2) has a concrete open-string-theoretical meaning of requiring ∇ to be massless from the viewpoint of open strings in Y via φ .*

2.3 Self-adjoint/Hermitian maps from an Azumaya/matrix manifold with a Hermitian fundamental module

When the complex vector bundle E over X is Hermitian, i.e. E is equipped with a smooth map

$$\langle \cdot, \cdot \rangle : E \times_X E \longrightarrow \mathbb{C}$$

that gives a Hermitian inner product on each fiber of E over X , one can require that the connection ∇ under consideration be unitary with respect to the Hermitian structure $\langle \cdot, \cdot \rangle$. This is a compatibility condition between the connection ∇ on E and the Hermitian metric $\langle \cdot, \cdot \rangle$ on E in the sense that the parallel transport defined by ∇ of a pair of elements in a fiber of E along a path in X would then preserve their inner product under $\langle \cdot, \cdot \rangle$.

Very naturally, one may ask:

- Q.** *Is there a condition on maps $\varphi : (X^{\text{Az}}, E) \rightarrow Y$ one can impose as well so that such φ can be thought of as being compatible with $\langle \cdot, \cdot \rangle$ in some sense?*

In this subsection, we answer this question affirmatively.

The adjoint φ^\dagger of φ with respect to $\langle \cdot, \cdot \rangle$ on E

The Hermitian structure $\langle \cdot, \cdot \rangle : E \times_X E \rightarrow \mathbb{C}$ induces an anti-linear isomorphism $E \simeq E^\vee$, $v \mapsto \langle v, \cdot \rangle$, as smooth complex vector bundles on X and hence an anti-linear anti-isomorphism $\text{End}_{\mathbb{C}}(E) \simeq \text{End}_{\mathbb{C}}(E^\vee)$, as \mathbb{C} -algebra bundles, given by $s \mapsto s^\dagger$ with $\langle s^\dagger(v), w \rangle = \langle v, s(w) \rangle$. With respect to a local trivialization of E by unitary frames with respect to $\langle \cdot, \cdot \rangle$, the adjoint s^\dagger of s is the transpose of the complex-conjugate of s .

Lemma 2.3.1. *Let $\varphi : (X^{\text{Az}}, E) \rightarrow Y$ be a differentiable map defined by a ring-homomorphism $\varphi^\# : C^\infty(Y) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ over $\mathbb{R} \subset \mathbb{C}$. With the above anti-linear anti-isomorphism, consider the specification*

$$\begin{aligned} \varphi^{\#\dagger} : C^\infty(Y) &\longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E)) \\ f &\longmapsto (\varphi^\#(f))^\dagger, \end{aligned}$$

where $(\varphi^\#(f))^\dagger$ is the adjoint of $\varphi^\#(f)$ with respect to $\langle \cdot, \cdot \rangle$. Then, $\varphi^{\#\dagger}$ is a ring-homomorphism over $\mathbb{R} \subset \mathbb{C}$.

Proof. Since $(m_1 m_2)^\dagger = m_2^\dagger m_1^\dagger$ for all $m_1, m_2 \in C^\infty(\text{End}_\mathbb{C}(E))$, φ^\sharp is a ring-anti-homomorphism by nature. However, $C^\infty(Y)$ is commutative; this renders φ^\sharp a ring-homomorphism. \square

It follows that $\varphi^\sharp : C^\infty(Y) \rightarrow C^\infty(\text{End}_\mathbb{C}(E))$ defines a differentiable map

$$\varphi^\dagger : (X^{\text{Az}}, E) \longrightarrow Y.$$

Definition 2.3.2. [adjoint/Hermitian conjugate of φ] The map $\varphi^\dagger : (X^{\text{Az}}, E) \rightarrow Y$ thus defined is called the *adjoint*, or synonymously the *Hermitian conjugate*, of φ with respect to the Hermitian structure $\langle \cdot, \cdot \rangle$ on E .

Lemma 2.3.3. [basic properties of φ^\dagger] (1) *The support of the graph of φ^\dagger and φ are identical, i.e. $\Gamma_{\varphi^\dagger} = \Gamma_\varphi$ as subschemes of $X \times Y$.* (2) *The graph of φ^\dagger and the graph of φ differ by an antiisomorphism, i.e. $\tilde{\mathcal{E}}_{\varphi^\dagger} \simeq \overline{\tilde{\mathcal{E}}_\varphi}$ as $\mathcal{O}_{X \times Y}^\mathbb{C}$ -modules.*

As a consequence, $\text{Im } \varphi = \text{Im } \varphi^\dagger$ as subschemes of Y , and $\varphi^\dagger_(\mathcal{E}) \simeq \overline{\varphi_*(\mathcal{E})}$ as $\mathcal{O}_Y^\mathbb{C}$ -modules.*

Proof. Statement (1) follows from the observation that, as $C^\infty(X)$ -algebras, $A_\varphi \simeq A_{\varphi^\dagger}$. Statement (2) follows from the observation that if two matrices m_1 and m_2 have their eigenvalues all real and $m_2 = m_1^\dagger$, then, their Jordan form can be made identical, and the fact that, by construction, $A_{\varphi^\dagger} = (A_\varphi)^\dagger$ as subalgebras of $C^\infty(\text{End}_\mathbb{C}(E))$. \square

Self-adjoint/Hermitian maps

Let E be equipped with a Hermitian structure $\langle \cdot, \cdot \rangle$.

Definition 2.3.4. [self-adjoint/Hermitian map] A differentiable map $\varphi : (X^{\text{Az}}, E) \rightarrow Y$ is called *self-adjoint*, or synonymously *Hermitian*, with respect to $\langle \cdot, \cdot \rangle$ if $\varphi^\dagger = \varphi$.

Lemma 2.3.5. [characterization by coordinate functions] *Let y^1, \dots, y^n be a set of coordinate functions of \mathbb{R}^n as a C^∞ -manifold and $\varphi : (X^{\text{Az}}, E) \rightarrow \mathbb{R}^n$ be a differentiable map. Then φ is self-adjoint if and only if each of $\varphi^\sharp(y^i)$, $i = 1, \dots, n$, is Hermitian.*

Proof. We only need to prove the if-part. This follows from the proof of Proposition 2.1.6, cf. [L-Y6: Theorem 3.2.1] (D(11.3.1)), reviewed in Sec. 4.1. In essence, as a consequence of the Malgrange Division Theorem, for any $f \in C^\infty(Y)$ and at the level of germs over X , $\varphi^\sharp(f)$ is expressible as a polynomial $P(\varphi^\sharp(y^1), \dots, \varphi^\sharp(y^n))$ with coefficients elements in a germ of smooth functions on X . The multi-degree of $P \leq (r-1, \dots, r-1)$ and the coefficients depend on f and location of the germ on X . Since the addition and the multiplication of commuting Hermitian matrices remain Hermitian, $\varphi^\sharp(f)$ is Hermitian, i.e. $(\varphi^\sharp(f))^\dagger = \varphi^\sharp(f)$. As $f \in C^\infty(Y)$ is arbitrary, this implies that $(\varphi^\sharp)^\dagger = \varphi^\sharp$. This concludes the proof. \square

Lemma 2.3.6. [X_φ generically reduced] *Let $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$ be a Hermitian map. Then X_φ is generically reduced.*

Proof. Commutating Hermitian matrices are simultaneously diagonalizable by a common unitary frame. Thus over each $p \in X$, the finite-dimensional \mathbb{R} -algebra $A_\varphi|_p$ has no nilpotent elements. The lemma follows. \square

Remark 2.3.7. [meaning of enhanced Lie-algebra-valued massless spectrum on D-brane world-volume] Recall from [Po3], [Po4], [Wi] that for coincident D-brane world-volume X of multiplicity r , the massless spectrum thereupon from excitations of oriented open strings consists of a $u(r)$ -valued gauge field and a $u(r)$ -valued scalar field. The former corresponds to $(E, \nabla) =$ a Hermitian vector bundle with a unitary connection, which describes the Spin-1 degrees of freedom on the D-brane while the latter corresponds a self-adjoint map $\varphi : (X^{\text{Az}}, E) \rightarrow Y$ that describes the Spin-0 degrees of freedom on the D-brane. This gives a precise interpretation of the related paragraphs in the above work of Polchinski and Witten. Here, the Lie algebra $u(r)$ from the unitary group $U(r)$ is identified with the fibers of the \mathbb{R} -vector subbundle $SAd(E, \langle, \rangle)$ of $End_{\mathbb{C}}(E)$ that consists of self-adjoint endomorphisms on fibers of (E, \langle, \rangle) . While $SAd(E)$ is *not* a bundle of rings, it makes sense to talk about *ring-homomorphisms* $\varphi^\sharp : C^\infty(Y) \rightarrow C^\infty(End_{\mathbb{C}})$ with values in $C^\infty(SAd(E, \langle, \rangle))$. They define precisely the self-adjoint maps in Definition 2.3.4.

Remark 2.3.8. [admissible Hermitian map] An admissible Hermitian map $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$ has the special property that over some open-dense subset $U \subset X$, $X_\varphi|_U$ is a covering space over U under the built-in map $X_\varphi|_U \rightarrow U$ and that $E|_U$ has an orthogonal decomposition, from the built-in isomorphism $\mathcal{E} \simeq \pi_{\varphi*}(\mathcal{O}_{X_\varphi} \mathcal{E})$, with each summand covariantly invariant under ∇ . They should be studied in more detail. Cf. Remark 3.2.5 and Remark 5.3.3.

Convention. For simplicity and a better focus on other issues that also occur, we'll assume for the rest of the work that the Admissibility Conditions (1) and (2) in Definition 2.1.1 apply to all over X .

3 The Dirac-Born-Infeld action for differentiable maps from Azumaya/matrix manifolds

Recall from Sec. 1 Issues (2) – (6) in the list one needs to understand to make sense of the formal expression of the Dirac-Born-Infeld action for stacked D-branes

$$S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla) \stackrel{\text{formally}}{=} -T_{m-1} \int_X \text{Tr} \left(e^{-\varphi^*(\Phi)} \sqrt{-\text{Det}_X(\varphi^*(g + B) + 2\pi\alpha' F_\nabla)} \right).$$

(2) [push-pull of tensor under φ]

The notion of *push-pulls under a differentiable map* φ from an Azumaya/matrix manifold with a fundamental module to a real manifold; cf. $\varphi^*(g + B)$.

(3) [determinant of 2-tensor on X]

The notion of *determinant* $\text{Det}_X(\cdots)$ in the current context once Issues (1) and (2) are resolved.

(4) [square root of matrix section]

Can we take a *square root of a matrix*? When the answer is Yes, is there a unique square root? If not, which one to choose? Extension of this to *matrix sections*?

(5) [*dilaton-field factor*]

How does *the factor* $e^{-\varphi^\sharp(\Phi)}$ influence the interpretation of the formal expression?

(6) [*real-valuedness*]

Is the expression *real*?

We now proceed to resolve all these issues (Sec. 3.1) and, hence, construct the Dirac-Born-Infeld action for admissible pairs (φ, ∇) (Sec. 3.2).

3.1 The resolution of issues toward defining the Dirac-Born-Infeld action

In this subsection we resolve Issues (2) – (6) in the list subsection by subsection.

3.1.1 The pull-back of tensors from the target space via commutative surrogates

We introduce the notion of ‘pull-push’ that works naturally for (φ, ∇) admissible and then discuss its basic properties and introduce its characteristic tensors. For the purpose of the current work, all the tensors on Y considered are of type $(0, \bullet)$, i.e. sections of $\otimes^\bullet T^*Y$.

The notion of ‘pull-push’

By itself, there is no problem to define the notion of tensors on an abstract “space” associated to a general (unital, associative but not necessarily commutative) ring. However, when compared with the definition for the same on a space associated to a commutative ring, the latter is a quotient of the former with additional relators arising from the commutativity relation of the commutative ring. Due to this, for a map between two spaces, with each associated to a ring,

$$f : \text{Space}(R) \longrightarrow \text{Space}(S)$$

defined through a ring-homomorphism $f^\sharp : S \rightarrow R$, with S commutative and R noncommutative, there is no canonical/natural notion of a pull-back f^* that takes tensors on $\text{Space}(S)$ to that on $\text{Space}(R)$. (See [L-Y4: Sec. 4.1] (D(11.1)), in particular, [ibidem: Example 4.1.20] for more details.) This is what happens in our situation for a map $\varphi : (X^{A_z}, E) \rightarrow Y$, defined by a ring-homomorphism $\varphi^\sharp : C^\infty(Y) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ over $\mathbb{R} \subset \mathbb{C}$.

On the other hand, while X^{A_z} is noncommutative, the surrogate X_φ of X^{A_z} associated to φ is commutative and fits into the following diagram that is canonically associated to φ

$$\begin{array}{ccc} X^{A_z} & & \\ \downarrow & \searrow \varphi & \\ X_\varphi & \xrightarrow{f_\varphi} & Y \\ \downarrow \pi_\varphi & & \\ X & & \end{array}$$

The existing notion of pull-back of tensors can be applied to define the pull-back $f_\varphi^* \Xi$ on X_φ of a tensor Ξ on Y under f_φ . Furthermore, when E is equipped with a connection ∇ and the pair (φ, ∇) is admissible, $f_\varphi^* \Xi$ can be naturally realized as a $\text{End}_{\mathbb{C}}(E)$ -valued tensor $\pi_{\varphi*} f_\varphi^* \Xi$ on X ; (cf. Sec. 2.2).

Definition 3.1.1.1. [pull-push from Y to X by φ] Let (φ, ∇) be admissible and Ξ a tensor on Y as above. We will denote $\pi_{\varphi*} f_{\varphi}^* \Xi$, which comes from the pull-push (i.e. first pulling back, then pushing forward) along the diagram associated to φ , by $\varphi^{\diamond} \Xi$ and call it the *pull-push* of the tensor Ξ on Y to X by φ .

In particular, for the 2-tensors metric g and B -field B on Y ,

$$\varphi^{\diamond}(g + B) := \pi_{\varphi*} f_{\varphi}^*(g + B)$$

is a well-defined $\text{End}_{\mathbb{C}}(E)$ -valued 2-tensor on X . This can then be added to a multiple of the curvature 2-tensor F_{∇} on X of the connection ∇ on E to give the $\text{End}_{\mathbb{C}}(E)$ -valued 2-tensor

$$\varphi^{\diamond}(g + B) + 2\pi\alpha' F_{\nabla}$$

on X . This is what we will interpret the object “ $\varphi^*(g + B) + 2\pi\alpha' F_{\nabla}$ ” in the formal expression of the Dirac-Born-Infeld action $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$, Sec. 1, in our context.

This resolves Issue (2) in the list.

Basic properties and the characteristic tensor of the pull-push

Lemma 3.1.1. [pull-push of symmetric tensor or alternating tensor] *Given an admissible pair (φ, ∇) as above, let Ξ be a tensor, say of degree l , on Y . Then:*

- (1) *If Ξ is a symmetric l -tensor on Y , then $\varphi^{\diamond} \Xi$ is an $\text{End}_{\mathbb{C}}(E)$ -valued symmetric l -tensor on X .*
- (2) *If Ξ is an alternating l -tensor (i.e. an l -form) on Y , then $\varphi^{\diamond} \Xi$ is an $\text{End}_{\mathbb{C}}(E)$ -valued alternating l -tensor (i.e. an $\text{End}_{\mathbb{C}}(E)$ -valued l -form) on X .*

Proof. The issue is local. Thus, for any $p \in X$, consider a small enough coordinate neighborhood U (with coordinate functions $\mathbf{x} = (x^1, \dots, x^m)$) of p such that $\varphi(U)$ lies in a local chart V of Y (with coordinate functions $\mathbf{y} = (y^1, \dots, y^n)$). Let $\Xi|_V = \sum_{i_1, \dots, i_l} \Xi_{i_1 \dots i_l} dy^{i_1} \otimes \dots \otimes dy^{i_l}$. Recall the connection D on $\text{End}_{\mathbb{C}}(E)$ canonically induced by ∇ on E . Denote $D_{\partial/\partial x^{\mu}}$, $\mu = 1, \dots, m$, by D_{μ} . Then, by definition,

$$\begin{aligned} (\varphi^{\diamond} \Xi)|_U &= \sum_{\mu_1, \dots, \mu_l=1}^m \left(\sum_{i_1, \dots, i_l=1}^n \varphi^{\sharp}(\Xi_{i_1 \dots i_l}) D_{\mu_1} \varphi^{\sharp}(y^{i_1}) \dots D_{\mu_l} \varphi^{\sharp}(y^{i_l}) \right) dx^{\mu_1} \otimes \dots \otimes dx^{\mu_l} \\ &=: \sum_{\mu_1, \dots, \mu_l=1}^m a_{\mu_1 \dots \mu_l} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_l}. \end{aligned}$$

Note that since (φ, ∇) is admissible, namely $D \cdot A_{\varphi} \subset A_{\varphi}$, all the elements

$$\varphi^{\sharp}(\Xi_{i_1 \dots i_l}), D_{\mu_1} \varphi^{\sharp}(y^{i_1}), \dots, D_{\mu_l} \varphi^{\sharp}(y^{i_l}) \in A_{\varphi}|_U \subset C^{\infty}(\text{End}_{\mathbb{C}}(E|_U)),$$

with $i_1, \dots, i_l = 1, \dots, n$, commute.

If Ξ is symmetric, then, for example,

$$\begin{aligned} a_{\mu_2 \mu_1 \mu_3 \dots \mu_l} &= \sum_{i_1, \dots, i_l=1}^n \varphi^{\sharp}(\Xi_{i_1 i_2 i_3 \dots i_l}) D_{\mu_2} \varphi^{\sharp}(y^{i_1}) D_{\mu_1} \varphi^{\sharp}(y^{i_2}) D_{\mu_3} \varphi^{\sharp}(y^{i_3}) \dots D_{\mu_l} \varphi^{\sharp}(y^{i_l}) \\ &= \sum_{i_1, \dots, i_l=1}^n \varphi^{\sharp}(\Xi_{i_1 i_2 i_3 \dots i_l}) D_{\mu_1} \varphi^{\sharp}(y^{i_2}) D_{\mu_2} \varphi^{\sharp}(y^{i_1}) D_{\mu_3} \varphi^{\sharp}(y^{i_3}) \dots D_{\mu_l} \varphi^{\sharp}(y^{i_l}) \\ &= \sum_{i_1, \dots, i_l=1}^n \varphi^{\sharp}(\Xi_{i_2 i_1 i_3 \dots i_l}) D_{\mu_1} \varphi^{\sharp}(y^{i_2}) D_{\mu_2} \varphi^{\sharp}(y^{i_1}) D_{\mu_3} \varphi^{\sharp}(y^{i_3}) \dots D_{\mu_l} \varphi^{\sharp}(y^{i_l}) \\ &= a_{\mu_1 \mu_2 \mu_3 \dots \mu_l}; \end{aligned}$$

and similarly for other exchanges of indices of $a_{\mu_1 \dots \mu_l}$. This proves that $\varphi^{\diamond} \Xi$ is symmetric and concludes Statement (1).

Statement (2) follows by a similar argument. □

For (φ, ∇) admissible, let Ξ be a tensor of degree l on Y . Then, since $\varphi^\diamond \Xi$ is A_φ -valued, in any local expression of $\varphi^\diamond \Xi$ in terms of local coordinate functions,

$$\varphi^\diamond \Xi|_U = \sum_{\mu_1, \dots, \mu_l=1}^m a_{\mu_1 \dots \mu_l} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_l},$$

the coefficients $a_{\mu_1 \dots \mu_l} \in A_\varphi|_U \subset C^\infty(\text{End}_{\mathbb{C}}(E|_U))$, $\mu_1, \dots, \mu_l = 1, \dots, m$, commute with each other and, hence, at each $p \in U$, can be simultaneously triangulated:

$$a_{\mu_1 \dots \mu_l} = G_p \cdot \begin{bmatrix} \lambda_{\mu_1 \dots \mu_l}^{(1)}(p) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_{\mu_1 \dots \mu_l}^{(r)}(p) \end{bmatrix} \cdot G_p^{-1},$$

where $G_p \in \text{Aut}(E|_p)$. The set of (ordinary) tensors at $p \in X$

$$\Lambda_{\varphi^\diamond \Xi}(p) := \{ \sum_{\mu_1, \dots, \mu_l} \lambda_{\mu_1 \dots \mu_l}^{(s)}(p) (dx^{\mu_1} \otimes \dots \otimes dx^{\mu_l})|_p \mid s = 1, \dots, r \} \subset (\otimes^l T^*X)|_p$$

is invariant under changes of coordinates on X and the local trivializations of E . As p varies, this defines a r -multi-section $\Lambda_{\varphi^\diamond \Xi}$ of $\otimes^l T^*X$.

Definition 3.1.1.2. [characteristic tensor of pull-push] With some abuse of the word ‘tensor’, the multi-section $\Lambda_{\varphi^\diamond \Xi}$ of $\otimes^l T^*X$ thus defined is called the *characteristic tensor* of the pull-push $\varphi^\diamond \Xi$ of an l -tensor Ξ on Y under an admissible map $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$.

The same proof as that of Lemma 3.1.1 gives:

Lemma 3.1.1.3. [characteristic tensor of pull-push of symmetric tensor or alternating tensor] *Continuing the setting in Lemma 3.1.1.*

- (1) *If Ξ is a symmetric tensor on Y , then $\Lambda_{\varphi^\diamond \Xi}$ is a symmetric multi-valued tensor on X .*
- (2) *If Ξ is an alternating tensor on Y , then $\Lambda_{\varphi^\diamond \Xi}$ is an alternating multi-valued tensor on X .*

Remark 3.1.1.4. [$\Lambda_{\varphi^\diamond \Xi}$ from aspect of C^∞ -algebraic geometry] As a subobject in the total space (denoted the same) of $\otimes^l T^*X$, $\Lambda_{\varphi^\diamond \Xi}$ is a C^∞ -subscheme of $\otimes^l T^*X$ that is algebraic and finite over X of relative length r . Its detail can be complicated. For the current notes, we use only its pointwise property over X in a few occasions.

Example 3.1.1.5. [pull-push of metric tensor on Y] Let (Y, g) be either a Riemannian manifold or a Lorentzian manifold and $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$ be an admissible map. Then $\varphi^\diamond g$ is an $\text{End}_{\mathbb{C}}(E)$ -valued symmetric 2-tensor on X .

Example 3.1.1.6. [pull-push of B -field on Y] Continuing Example 3.1.1.5. Let B be a 2-form on Y . Then $\varphi^\diamond B$ is an $\text{End}_{\mathbb{C}}(E)$ -valued 2-form on X .

3.1.2 D-brane world-volume with constant induced-metric signature

For a simple D-brane moving in a space-time Y , by definition it sweeps out a D-brane world-volume that has a Lorentzian induced metric. This gives the simplest picture of D-brane world-volume: A Lorentzian submanifold (with a Chan-Paton bundle with a connection, ...) in a space-time Y . Now that we generalize the notion of a submanifold to the notion of differentiable map $\varphi : (X^{Az}, E) \rightarrow Y$, a question arises immediately:

Q. *Given a Lorentzian manifold (Y, g) , in what sense can one say that $\varphi : (X^{Az}, E) \rightarrow Y$ is Lorentzian (or equivalently, timelike), or spacelike, or null?*

A geometrically reasonable, though naive, approach is to consider the C^∞ -subscheme

$$\Gamma_\varphi := \text{Supp}(\tilde{\mathcal{E}}_\varphi) \subset X \times Y,$$

which is canonically isomorphic to the surrogate X_φ , and look at the restriction $(pr_Y^*g)|_{\Gamma_\varphi}$ or $(pr_Y^*g)|_{(\Gamma_\varphi)_{red}}$. Here, $pr_Y : X \times Y \rightarrow Y$ is the projection map. In [L-Y4: Sec. 6.3] (D(11.1)), we took such an approach to define notions such as ‘*Lagrangian maps*’ to a symplectic manifold or ‘*special Lagrangian maps*’ to a Calabi-Yau manifold. The setting is independent of the connection ∇ on E .

However, in the course of understanding the Dirac-Born-Infeld action in our context, it turns out that, for an admissible (φ, ∇) , the following definition is algebraically and technically more natural: (cf. Lemma 3.1.4.7)

Definition 3.1.2.1. [Lorentzian/timelike, spacelike, null map] Let (Y, g) be a Lorentzian manifold (of signature $(-, +, \dots, +)$). An admissible map $\varphi : (X^{Az}, E; \nabla) \rightarrow Y$ is said to be *Lorentzian*, or equivalently *timelike*, (resp. *spacelike*, *null*) if for any $p \in X$, each symmetric 2-tensor in the characteristic tensor-set $\Lambda_{\varphi \circ g}(p)$ (cf. Definition 3.1.1.2) defines a Minkowskian (resp. Euclidean, degenerate with signature $(0, +, \dots, +)$) inner product on $T_p X$.

Definition 3.1.2.2. [Riemannian map] Let (Y, g) be a Riemannian manifold. An admissible map $\varphi : (X^{Az}, E; \nabla) \rightarrow Y$ is said to be *Riemannian* if for any $p \in X$, each symmetric 2-tensor in the characteristic tensor-set $\Lambda_{\varphi \circ g}(p)$ defines a Euclidean inner product on $T_p X$.

The relation, or discrepancy, between the setting following [L-Y4: Sec. 6.3] (D(11.1)) and the setting in the above two definitions should be investigated further.

3.1.3 From determinant Det to symmetrized determinant $SymDet$

For comparison and motivation, we review first the defining properties of the determinant function Det over a commutative ring and then generalize it to the notion of symmetrized determinant $SymDet$ in the noncommutative case. This is then applied to define the notion of symmetrized determinant of an $End_{\mathbb{C}}(E)$ -valued 2-tensor on X .

Convention [oriented manifold and compatible system of coordinate functions] To have a globally well-defined volumed form, rather than just a density or measure, for the rest of the notes, we assume:

- Both X and Y are oriented manifolds.

- Whenever a system of local coordinate functions are chosen, e.g. $\mathbf{x} = (x^1, \dots, x^m)$ for some local chart $U \subset X$ and $\mathbf{y} = (y^1, \dots, y^n)$ for some local chart $V \subset Y$, the order of these functions is chosen so that $dx^1 \wedge \dots \wedge dx^m$ specifies the orientation on U and $dy^1 \wedge \dots \wedge dy^n$ specifies the orientation on V .

The determinant function Det over a commutative ring

We summarize the defining properties of the determinant function Det over a commutative ring into the following two definitions and theorem. Readers are referred to [H-K: Chapter 5] for details.

Definition 3.1.3.1. [multi-linear alternating function on matrices] Let R be a commutative ring with the identity element 1, $M_{l \times l}(R)$ be the ring of $l \times l$ matrices with entries in R . A function

$$f : M_{l \times l}(R) \longrightarrow R$$

is called *l-linear alternating* if

- [*l-linear*] For each i , $1 \leq i \leq l$, f is an R -linear function of the i -th row when the other $(l - 1)$ rows are held fixed.
- [*alternating*] The following two conditions are satisfied:
 - $f(m) = 0$ whenever two rows of $m \in M_{l \times l}(R)$ are equal.
 - If m' is obtained from $m \in M_{l \times l}(R)$ by interchanging two rows of m , then $f(m') = -f(m)$.

Definition 3.1.3.2. [determinant function] Continuing the setting of Definition 3.1.3.1. A function $f : M_{l \times l}(R) \rightarrow R$ is called a *determinant function* if f is l -linear, alternating, and $f(Id_{l \times l}) = 1$. Here, $Id_{l \times l}$ is the identity matrix in $M_{l \times l}(R)$.

Theorem 3.1.3.3. [existence and uniqueness of determinant function] Continuing the setting of Definition 3.1.3.1. *There exists a unique determinant function $M_{l \times l}(R) \rightarrow R$. Denote this function by Det . Then, for $m = (m_{ij})_{ij} \in M_{l \times l}(R)$,*

$$Det(m) = \sum_{\sigma \in Sym_l} (-1)^\sigma m_{1\sigma(1)} \cdots m_{l\sigma(l)}.$$

Here, Sym_l is the permutation group on l -many letters, and $(-1)^\sigma = 1$ (resp. -1) if σ is an even (resp. odd) permutation.

With the above review, the question now is:

- Q.** *Can the above functorial definition of the determinant function Det be generalized to the case where R is noncommutative?*

The symmetrized determinant over a noncommutative ring

Let R be an (associative, unital) ring that is not necessarily commutative. Our goal now is to generalize the determinant function Det above for R commutative to the current case. When R is noncommutative, one learns from experience that it is very restrictive to require a function $f : M_{l \times l}(R) \rightarrow R$ to be multi- R -linear and it is more practical to demand only that $f : M_{l \times l}(R) \rightarrow R$ be multi- $C(R)$ -linear, where $C(R)$ is the center of R . This motivates the following definition:

Definition 3.1.3.4. [multi-central-linear alternating function on matrices] Let R be a (unital associative) ring with the identity element 1. Denote by $C(R)$ the center of R . $M_{l \times l}(R)$ be the ring of $l \times l$ matrices with entries in R . A function

$$f : M_{l \times l}(R) \longrightarrow R$$

is called *l-central linear alternating* if

- [*l-central linear*] For each i , $1 \leq i \leq l$, f is an $C(R)$ -linear function of the i -th row when the other $(l - 1)$ rows are held fixed.
- [*alternating*] The following two conditions are satisfied:
 - $f(m) = 0$ whenever two rows of $m \in M_{l \times l}(R)$ are equal.
 - If m' is obtained from $m \in M_{l \times l}(R)$ by interchanging two rows of m , then $f(m') = -f(m)$.

Definition 3.1.3.5. [determinant function – noncommutative case] Continuing the setting of Definition 3.1.3.4. A function $f : M_{l \times l}(R) \rightarrow R$ is called a *determinant function* if f is l -central linear, alternating, and $f(Id_{l \times l}) = 1$. Here, $Id_{l \times l}$ is the identity matrix in $M_{l \times l}(R)$.

The following definition and lemma answer the existence part of a determinant function in the noncommutative case:

Definition 3.1.3.6. [symmetrized determinant] Let R be a (unital associative) ring with the identity element 1 and $M_{l \times l}(R)$ be the ring of $l \times l$ matrices with entries in R . Define the *symmetrized determinant function*

$$SymDet : M_{l \times l}(R) \longrightarrow R$$

by the assignment to $m = (m_{ij})_{ij} \in M_{l \times l}(R)$ the following element in R

$$SymDet(m) := \sum_{\sigma \in Sym_l} (-1)^\sigma m_{1\sigma(1)} \odot \cdots \odot m_{l\sigma(l)},$$

where

$$r_1 \odot \cdots \odot r_l := \frac{1}{l!} \sum_{\sigma' \in Sym_l} r_{\sigma'(1)} \cdots r_{\sigma'(l)}$$

is the *symmetrized product* of $r_1, \dots, r_l \in R$. Here, Sym_l is the permutation group on l letters.

The lemma below justifies the name:

Lemma 3.1.3.7. [*SymDet* as generalization of *Det*] Continuing the setting in Definition 3.1.3.6. The correspondence $SymDet : M_{l \times l}(R) \rightarrow R$ is a determinant function. Furthermore, when R is commutative, $SymDet$ and Det coincide.

Proof. That $SymDet$ is l -central linear and that $SymDet(Id_{l \times l}) = 1$ are immediate. To show that $SymDet$ is alternating, observe that

$$r_1 \odot \cdots \odot r_l = r_{\sigma'(1)} \odot \cdots \odot r_{\sigma'(l)}$$

for any $\sigma' \in Sym_l$. Consequently, the proof that $SymDet$ is alternating follows exactly the same proof that Det is alternating since in the latter case only the commutativity of factors in the l -products in the expansion of $Det(\cdot)$ and the sign $(-1)^\sigma = \pm 1$, $\sigma \in Sym_l$, before the l -products are used in the proof. That $SymDet$ and Det coincide when R is commutative is clear by the definition of $SymDet$. □

Lemma 3.1.3.8. [*SymDet* in terms of *Det*] Continuing the setting in Definition 3.1.3.6. Let

$$m = \begin{bmatrix} m_{(1)} \\ \vdots \\ m_{(l)} \end{bmatrix} = [m_{(1)}^\top, \dots, m_{(l)}^\top]^\top$$

be the presentation of an $l \times l$ matrix m in terms of its row vectors $m_{(1)}, \dots, m_{(l)}$. Here, $[\cdot]^\top$ denotes the transpose of a matrix $[\cdot]$. Then,

$$SymDet(m) = \frac{1}{l!} \sum_{\sigma \in Sym_l} (-1)^\sigma Det([m_{\sigma(1)}^\top, \dots, m_{\sigma(l)}^\top]^\top),$$

where we define $Det(m) := \sum_{\sigma \in Sym_l} (-1)^\sigma m_{1\sigma(1)} \cdots m_{l\sigma(l)}$.

Proof. This follows directly from the definition of $SymDet$ and Det . □

Caution that for R noncommutative, Det as defined is, in general, not a determinant function in the sense of Definition 3.1.3.5

Remark 3.1.3.9. [*uniqueness*] It is not clear to us whether the symmetrized determinant $SymDet$ is the only determinat function (in the sense of Definition 3.1.3.5) that can be defined on $M_{l \times l}(R)$ for R noncommutative.

Remark 3.1.3.10. [*on the altered ring $(R, +, \odot)$*] Caution that we directly define the symmetrized product $r_1 \odot \cdots \odot r_l$ for an l -tuple (r_1, \dots, r_l) of elements in R in the definition of $SymDet$, rather than building it up through a binary operation. This is all we need and used. The ring $(R, +, \odot)$ altered from the original R is commutative and unital, but in general no longer associative: For example, the three products

$$r_1 \odot r_2 \odot r_3, \quad (r_1 \odot r_2) \odot r_3, \quad r_1 \odot (r_2 \odot r_3)$$

in general are all different. In particular, a property or an identity related to $Det(\cdot)$ that relies only on the commutativity of the underlying ring automatically passes over to $SymDet(\cdot)$ in the noncommutative case, while a property or an identity related to $Det(\cdot)$ that involves also the associativity of the underlying ring either fails or requires to be checked independently. For this reason, one does not have a simple formula that expresses $SymDet(m)$ in terms of a row or column of m and the corresponding $(l-1) \times (l-1)$ minors of m .

The symmetrized determinant $SymDet_X(\Xi)$ of an $End_{\mathbb{C}}(E)$ -valued 2-tensor Ξ on X

We are now ready to address the notion of ‘determinant’ that appears in the formal expression of the Dirac-Born-Infeld action $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$, Sec. 1.

Ansatz 3.1.3.11. [$SymDet$ in the Dirac-Born-Infeld action] We interpret the determinant that appears in the formal expression of the Dirac-Born-Infeld action $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$, Sec. 1, as the *symmetrized determinant* $SymDet$ that applies to an $End_{\mathbb{C}}(E)$ -valued 2-tensor Ξ on X .

We now explain the details of this determinant in our context of D-branes.

Let $\Xi \in C^\infty((T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E))$ be an $End_{\mathbb{C}}(E)$ -valued 2-tensor on X . Locally on a coordinate chart $U \subset X$ (with coordinates (x^1, \dots, x^m)) Ξ has an expression of the form

$$\sum_{\mu, \nu=1}^m \Xi_{\mu\nu} dx^\mu \otimes dx^\nu,$$

with the coefficients in the (unital, associative) endomorphism ring (with the identity element $Id_{r \times r}$):

$$\Xi_{\mu\nu} \in C^\infty(End_{\mathbb{C}}(E|_U)).$$

The local coefficients form a $m \times m$ matrix, with the (μ, ν) -entry $\Xi_{\mu\nu}$:

$$\hat{\Xi}_U := (\Xi_{\mu\nu})_{\mu\nu} \in M_{m \times m}(C^\infty(End_{\mathbb{C}}(E|_U))).$$

Definition 3.1.3.12. [symmetrized determinant of $End_{\mathbb{C}}(E)$ -valued 2-tensor] With the notation from above, the *symmetrized determinant*

$$SymDet_X(\Xi) \in C^\infty((\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E))$$

of the $End_{\mathbb{C}}(E)$ -valued 2-tensor Ξ is defined to be the $End_{\mathbb{C}}(E)$ -valued $2m$ -tensor on X , locally defined by

$$SymDet(\hat{\Xi}_U)(dx^1 \wedge \dots \wedge dx^m)^{\otimes 2}$$

on a coordinate chart $U \subset X$ with coordinate functions (x^1, \dots, x^m) .

Lemma 3.1.3.13. [well-definedness of $SymDet(\Xi)$] *The symmetrized determinant $SymDet(\Xi)$ of Ξ , as defined in Definition 3.1.3.12, is well-defined.*

Proof. We only need to show that the local expressions of $SymDet(\Xi)$ transform from one chart to another under the local coordinate transformation on X and the accompanying local transition on E . This is a standard computation. Let U_α and U_β be overlapping local charts on X with coordinates $\mathbf{x}_\alpha := (x_\alpha^1, \dots, x_\alpha^m)$ and $\mathbf{x}_\beta := (x_\beta^1, \dots, x_\beta^m)$ respectively. Let

$$\begin{aligned} \phi_{\alpha\beta} : (E|_{U_\alpha})|_{U_\alpha \cap U_\beta} &\longrightarrow (E|_{U_\beta})|_{U_\beta \cap U_\alpha} \\ (\mathbf{x}_\alpha, \mathbf{v}_\alpha) &\longmapsto (\mathbf{x}_\beta, \mathbf{v}_\beta) = (h_{\alpha\beta}(\mathbf{x}_\alpha), \hat{h}_{\alpha\beta}(\mathbf{x}_\alpha)(\mathbf{v}_\alpha)) \end{aligned}$$

be the transition map. Then the transition of local expressions of Ξ is given by

$$\begin{aligned} \phi_{\alpha\beta}^*(\Xi_\beta) &= \phi_{\alpha\beta}^*(\sum_{\mu', \nu'} \Xi_{\beta, \mu' \nu'}(\mathbf{x}_\beta) dx_\beta^{\mu'} \otimes dx_\beta^{\nu'}) \\ &= \sum_{\mu, \nu} \left(\sum_{\mu', \nu'} \hat{h}_{\alpha\beta}(\mathbf{x}_\alpha)^{-1} \Xi_{\beta, \mu' \nu'}(h_{\alpha\beta}(\mathbf{x}_\alpha)) \hat{h}_{\alpha\beta}(\mathbf{x}_\alpha) \frac{\partial h_{\alpha\beta}^{\mu'}(\mathbf{x}_\alpha)}{\partial x_\alpha^\mu} \frac{\partial h_{\alpha\beta}^{\nu'}(\mathbf{x}_\alpha)}{\partial x_\alpha^\nu} \right) dx_\alpha^\mu \otimes dx_\alpha^\nu \\ &= \sum_{\mu, \nu} \Xi_{\alpha, \mu\nu}(\mathbf{x}_\alpha) dx_\alpha^\mu \otimes dx_\alpha^\nu. \end{aligned}$$

In terms of the local coefficient matrix of $End_{\mathbb{C}}(E)$ -valued 2-tensors on X , this says that

$$\hat{\Xi}_{U_\alpha} = \frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha}^T Ad_{\hat{h}_{\alpha\beta}^{-1}}(\hat{\Xi}_{U_\beta}) \frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha},$$

where the Adjoint $Ad_{\hat{h}_{\alpha\beta}^{-1}}$ acts on the $End_{\mathbb{C}}(E|_{U_\alpha \cap U_\beta})$ -valued entries of $\hat{\Xi}_{U_\beta}$, and $\frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha}$ is the $m \times m$ Jacobian matrix of $h_{\alpha\beta}$ with $\frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha}^T$ its transpose.

It follows that

$$\begin{aligned} &SymDet(\hat{\Xi}_{U_\alpha})(dx^1 \wedge \dots \wedge dx^m)^{\otimes 2} \\ &= SymDet\left(\frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha}^T Ad_{\hat{h}_{\alpha\beta}^{-1}}(\hat{\Xi}_{U_\beta}) \frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha}\right)(dx^1 \wedge \dots \wedge dx^m)^{\otimes 2} \\ &= Ad_{\hat{h}_{\alpha\beta}^{-1}}\left(SymDet(\hat{\Xi}_{U_\beta})\right)\left(Det\left(\frac{\partial \mathbf{x}_\beta}{\partial \mathbf{x}_\alpha}\right)\right)^2(dx^1 \wedge \dots \wedge dx^m)^{\otimes 2} \\ &= \phi_{\alpha\beta}^*\left(SymDet(\hat{\Xi}_{U_\beta})(dx_\beta^1 \wedge \dots \wedge dx_\beta^m)^{\otimes 2}\right). \end{aligned}$$

In other words, the collection

$$\left\{ SymDet(\hat{\Xi}_{U_\alpha})(dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m)^{\otimes 2} \right\}_\alpha$$

of local sections of $(\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E)$ glue to a global section of $(\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E)$. This concludes the lemma. \square

This resolves Issue (3) in the list.

3.1.4 Square roots of sections of $(\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E)$

We give first a general study of square roots of matrices in $M_{r \times r}(\mathbb{C})$ and then apply it to understand the square roots of sections of $(\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E)$.

Square roots of matrices in $M_{r \times r}(\mathbb{C})$

Note first that for an arbitrary $r \times r$ matrix $m \in M_{r \times r}(\mathbb{C})$, there may not be an $m' \in M_{r \times r}(\mathbb{C})$ that satisfies $(m')^2 = m$. In other words, a square root of m may not exist. This is illustrated by the following example:

Example 3.1.4.1. [matrix with no square root] Let $r \geq 2$ and $m \in M_{r \times r}(\mathbb{C})$ be a nilpotent matrix of nilpotency r . If a square root m' of m exists, then m' must also be nilpotent, of nilpotency $\leq r$. But this implies in turn that the nilpotency of $m = (m')^2$ must be strictly less than r , which is a contradiction.

On the other hand, one has the following affirmative situation:

Example 3.1.4.2. [neighborhood of diagonalizable matrices with nonzero eigenvalues] First notice that a diagonalizable matrix in $M_{r \times r}(\mathbb{C})$ with all its eigenvalues nonzero has 2^r -many square roots. Using the Implicit Function Theorem, one can show then that any matrix in a small enough neighborhood of such a matrix in $M_{r \times r}(\mathbb{C})$ has also 2^r -many square roots.

Indeed, motivated by how a commutative subalgebra of $M_{r \times r}(\mathbb{C})$ is canonically a C^∞ -ring (cf. [L-Y6: Sec. 2] (D(11.3.1))), one can prove a stronger result than Example 3.1.4.2:

Lemma 3.1.4.3. [existence of 2^r -many square roots of invertible matrix] Let $GL_r(\mathbb{C}) = \{m \mid \det m \neq 0\}$ be the open subset of $M_{r \times r}(\mathbb{C})$ that consists of invertible $r \times r$ matrices, with the subset topology from the isomorphism $M_{r \times r}(\mathbb{C}) \simeq \mathbb{C}^{\oplus r^2}$ as \mathbb{C} vector spaces. Then,

$$\begin{aligned} \Upsilon : GL_r(\mathbb{C}) &\longrightarrow GL_r(\mathbb{C}) \\ m' &\longmapsto (m')^2 \end{aligned}$$

is a covering map of degree 2^r . It follows that for $m \in GL_r(\mathbb{C})$, m has exactly 2^r -many distinct square roots.

Proof. Let $m \in GL_r(\mathbb{C})$. We construct first 2^r -many local inverses $\Upsilon^{-1}(m)$ to Υ at m as follows. Let $v : \mathbb{C} \rightarrow \mathbb{C}$ be the map $z \mapsto z^2$, $z_0 \in \mathbb{C} - \{0\}$, and \sqrt{v} be either of the square root of v , defined and analytic on a simply connected region $\Omega \subset \mathbb{C} - \{0\}$ that contains z_0 . Consider the Taylor expansion of \sqrt{v} at z_0 for z with $|z - z_0| \leq |z_0|$:

$$\sqrt{v}(z) = \sum_{l=0}^{r-1} \frac{(-1)^{l-1}(2l-3)!!}{l! \cdot 2^l} \frac{\sqrt{z_0}}{z_0^l} \cdot (z - z_0)^l + O((z - z_0)^r),$$

where $(2l-3)!! := \prod_{i=1}^{l-1} (2i-1)$ for $l \geq 2$, and $(-3)!! = -1$, $(-1)!! = 1$ by convention. Let $m = G_m J_m G_m^{-1}$, where $G_m \in GL_r(\mathbb{C})$ and J_m is the Jordan form of m . Then $J_m = D_m + N_m$, where D_m is diagonal and N_m is upper triangular and nilpotent, such that $D_m N_m = N_m D_m$. In terms of this,

$$\sqrt{v}(m) = G_m \left(\sum_{l=0}^{r-1} \frac{(-1)^{l-1}(2l-3)!!}{l! \cdot 2^l} \frac{\sqrt{D_m}}{D_m^l} N_m^l \right) G_m^{-1}.$$

This reduce the problem of defining $\sqrt{v}(m)$ to the existence of $\sqrt{D_m}$. The latter holds, since D_m is diagonal, and has 2^r -many choices. This says that m has 2^r -many inverses, counted with multiplicity, under Υ .

Since $m \in GL_r(\mathbb{C})$, all the diagonal entries in the diagonal matrix D_m is non-zero. Thus, all these inverses must be simple (i.e. distinct of multiplicity 1). It follows that the construction can be extended to a small enough neighborhood of $m \in GL_r(\mathbb{C})$ to define 2^r -many distinct local inverses to γ around m . This concludes the lemma. \square

Definition 3.1.4.4. [principal square root] (1) For a diagonal matrix $D \in M_{r \times r}(\mathbb{C})$ with all the diagonal entries positive, we define the *principal square root* of D , in notation \sqrt{D} , to be the unique square root of D that has all the diagonal entries positive as well. (2) For $m \in GL_r(\mathbb{C})$ that lies in a small enough neighborhood of the conjugacy class of a diagonal matrix in $GL_r(\mathbb{C})$ with all the diagonal entries positive, we define the *principal square root* of m , in notation \sqrt{m} , to be the unique square root of m that has all its eigenvalues λ_i satisfying $\operatorname{Re} \lambda_i > 0$.

Principal square root of elements in A_φ .

Let $\varphi : (X^{\text{Az}}, E) \rightarrow Y$ be a differentiable map, defined by a ring-homomorphism $\varphi^\# : C^\infty(Y) \rightarrow C^\infty(\operatorname{End}_{\mathbb{C}}(E))$ over $\mathbb{R} \subset \mathbb{C}$. As an intermediate step, consider the notion of ‘square roots’ of elements in $A_\varphi := C^\infty(X) \langle \operatorname{Im}(\varphi^\#) \rangle$. Through the built-in inclusion $A_\varphi \subset C^\infty(\operatorname{End}_{\mathbb{C}}(E))$ and the study of the previous theme ‘Square roots of matrices in $M_{r \times r}(\mathbb{C})$ ’, one learns that an element $s \in A_\varphi$ may not have a square root in A_φ ; namely, there may be no element $s' \in A_\varphi$ such that $(s')^2 = s$. However, from the proof of Lemma 3.1.4.3, one learns that fiberwise over p , if a principal square root of $s(p) \in \operatorname{End}_{\mathbb{C}}(E|_p)$ exists, it comes from the C^∞ -ring structure of $A_\varphi|_p$. It follows that for $s \in A_\varphi$, if there is an $s' \in C^\infty(\operatorname{End}_{\mathbb{C}}(E))$ such that for all $p \in X$, $s'(p)$ is the principal square root of s , then s' must lie in A_φ .

Definition 3.1.4.5. [principal square root of element in A_φ] An $s' \in A_\varphi$ is the *principal square root* of $s \in A_\varphi$ if for all $p \in X$, $s'(p)$ is the principal square root of $s(p)$.

Lemma 3.1.4.6. [criterion for existence of principal square root] Let $s \in A_\varphi \subset \operatorname{End}_{\mathbb{C}}(E)$. Then, the *principal square root* \sqrt{s} of s exists in A_φ if and only if, for all $p \in X$, the *principal square root* $\sqrt{s(p)} \in \operatorname{End}_{\mathbb{C}}(E|_p)$ exists.

Proof. We only need to show the if-part. For that, one only needs to show that the correspondence $X \rightarrow \operatorname{End}_{\mathbb{C}}(E)$ with $p \mapsto \sqrt{s(p)}$ is smooth. The latter follows from the observation that in the current situation $s \in C^\infty(\operatorname{Aut}_{\mathbb{C}}(E)) \subset C^\infty(\operatorname{End}_{\mathbb{C}}(E))$ and that the square map $\operatorname{Aut}_{\mathbb{C}}(E) \rightarrow \operatorname{Aut}_{\mathbb{C}}(E)$, $h \mapsto h^2$, over X is a smooth covering map and, hence, its local inverse, which takes in particular s to \sqrt{s} , must be a diffeomorphism. \square

Square roots of sections of $(\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{C}}(E)$

It follows from the previous theme ‘Square roots of matrices in $M_{r \times r}(\mathbb{C})$ ’ that for a general section s of $(\bigwedge^m T^*X)^{\otimes 2} \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{C}}(E)$, the square root of s is only a rational multi-section of $\bigwedge^m T^*X \otimes_{\mathbb{R}} \operatorname{End}_{\mathbb{C}}(E)$, defined on an open subset of X .

Lemma 3.1.4.7. [principal square root of symmetrized determinant of pull-push of metric tensor] (1) Let (Y, g) be a Lorentzian n -manifold and $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$ be a Lorentzian admissible map. Then

$$- \text{SymDet}_X(\varphi^\diamond g) \in C^\infty((\wedge^m T^* X)^{\otimes 2} \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E))$$

has a well-defined principal square root

$$\sqrt{-\text{SymDet}_X(\varphi^\diamond g)} \in C^\infty(\wedge^m T^* X \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)).$$

(2) Let either (Y, g) be a Lorentzian n -manifold and $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$ be a spacelike admissible map, or (Y, g) be a Riemannian n -manifold and $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow Y$ be a Riemannian admissible map. Then

$$\text{SymDet}_X(\varphi^\diamond g) \in C^\infty((\wedge^m T^* X)^{\otimes 2} \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E))$$

has a well-defined principal square root

$$\sqrt{\text{SymDet}_X(\varphi^\diamond g)} \in C^\infty(\wedge^m T^* X \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)).$$

Proof. Let $\text{Aut}_{\mathbb{C}}(E) \subset \text{End}_{\mathbb{C}}(E)$ be the automorphism bundle of E . Then it follows from Lemma 3.1.4.3 that the map $\text{Aut}_{\mathbb{C}}(E) \rightarrow \text{Aut}_{\mathbb{C}}(E)$, $h \mapsto h^2$, over X is a covering map of degree 2^r , where r is the rank of E as a complex vector bundle over X . It follows that as long as a section s in $(\wedge^m T^* X)^{\otimes 2} \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)$ lies in the open subset $(\wedge^m T^* X)^{\otimes 2} \otimes_{\mathbb{R}} \text{Aut}_{\mathbb{C}}(E)$ and for each point $p \in X$, the principal square root $\sqrt{s(p)}$ exists, then \sqrt{s} exists as a smooth section of $\wedge^m T^* X \otimes_{\mathbb{R}} \text{Aut}_{\mathbb{C}}(E) \subset \wedge^m T^* X \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)$.

For Statement (1), for each $p \in X$, let

$$\lambda^{(1)}, \dots, \lambda^{(r)} \in \Lambda_{\varphi^\diamond g}(p)$$

give the characteristic tensor of $\varphi^\diamond g$ over p . Counted with multiplicity, they define r -many inner products on $T_p X$. By construction,

$$\text{SymDet}((\varphi^\diamond g)(p)) = \text{Det}((\varphi^\diamond g)(p)) = G_p \cdot \begin{bmatrix} \text{Det}(\lambda^{(1)}) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \text{Det}(\lambda^{(r)}) \end{bmatrix} \cdot G_p^{-1},$$

for some $G_p \in \text{Aut}(E|_p)$. Since φ is Lorentzian, each λ^i defines a (non-degenerate) Minkowskian inner product on $T_p X$, for $i = 1, \dots, r$. It follows that the proof of Lemma 3.1.4.3 that

$$- \text{SymDet}((\varphi^\diamond g)(p)) = - \text{Det}((\varphi^\diamond g)(p)) = G_p \cdot \begin{bmatrix} -\text{Det}(\lambda^{(1)}) & -* & -* \\ 0 & -\ddots & -* \\ 0 & 0 & -\text{Det}(\lambda^{(r)}) \end{bmatrix} \cdot G_p^{-1}$$

admits a principal square root $\sqrt{-\text{SymDet}(\varphi^\diamond g)}$ of the form

$$\sqrt{-\text{SymDet}((\varphi^\diamond g)(p))} = G_p \cdot \begin{bmatrix} \sqrt{-\text{Det}(\lambda^{(1)})} & *' & *' \\ 0 & \ddots & *' \\ 0 & 0 & \sqrt{-\text{Det}(\lambda^{(r)})} \end{bmatrix} \cdot G_p^{-1}.$$

This proves Statement (1)

Statement (2) is proved by a similar argument.

□

It follows that, for a Lorentzian map $\varphi : (X^{\text{Az}}, E; \nabla) \rightarrow (Y, g, B)$,

- If B and F_∇ are small, the $\text{End}_{\mathbb{C}}(E)$ -valued 2-tensor $-\text{SymDet}_X(\varphi^\diamond(g+B) + 2\pi\alpha'F_\nabla)$, now regarded as from a deformation of $-\text{SymDet}_X(\varphi^\diamond g)$, has a well-defined principal square root $\sqrt{-\text{SymDet}(\varphi^\diamond(g+B) + 2\pi\alpha'F_\nabla)}$.

Similarly, for the other two situations.

This resolves Issue (4) in the list.

Remark 3.1.4.8. [Where the tensors take their value] Let Ξ be a 2-tensor on Y . By construction, both $\varphi^\diamond \Xi$ and, hence, its symmetrized determinant $\text{SymDet}_X(\varphi^\diamond \Xi)$ are A_φ -valued tensors on X . It follows from Lemma 3.1.4.6 and the construction of $\sqrt{\pm \text{SymDet}_X(\varphi^\diamond \Xi)}$ that if the principal square root $\sqrt{\pm \text{SymDet}_X(\varphi^\diamond \Xi)}$ exists as an $\text{End}_{\mathbb{C}}(E)$ -valued tensor on X , then it must be indeed A_φ -valued. When in addition F_∇ is taken into account, in general $\varphi^\diamond \Xi + 2\pi\alpha'F_\nabla$, and, hence, $\text{SymDet}_X(\varphi^\diamond \Xi + 2\pi\alpha'F_\nabla)$ and $\sqrt{\pm \text{SymDet}_X(\varphi^\diamond \Xi + 2\pi\alpha'F_\nabla)}$ (if defined) are only $\text{End}_{\mathbb{C}}(E)$ -valued. This applies when $\Xi = g, B$, or $g+B$.

3.1.5 The factor from the dilaton field Φ on the target space(-time)

The dilaton field Φ is a scalar field on Y . We will take Φ as smooth, i.e. $\Phi \in C^\infty(Y)$. Then, by the definition of pull-push under $\varphi : (X^{\text{Az}}, E) \rightarrow Y$,

$$\varphi^\diamond \Phi = \varphi^\sharp(\Phi) \quad \text{and} \quad e^{-\varphi^\diamond \Phi} = \varphi^\sharp(e^{-\Phi}) = e^{-\varphi^\sharp(\Phi)} \in A_\varphi.$$

Here, $e^{-\varphi^\sharp(\Phi)}$ is defined through the C^∞ -ring structure of A_φ . As noted in Remark 3.1.4.8, $\sqrt{-\text{SymDet}_X(\varphi^\diamond(g+B))}$ is A_φ -valued, and, hence,

- The factor $e^{-\varphi^\diamond \Phi}$ and the principal square root $\sqrt{-\text{SymDet}_X(\varphi^\diamond(g+B))}$ commute.

Once the gauge curvature F_∇ is also taken into account, $e^{-\varphi^\diamond \Phi}$ and $\sqrt{-\text{SymDet}_X(\varphi^\diamond(g+B) + 2\pi\alpha'F_\nabla)}$ may not commute for a general F_∇ .

Lemma 3.1.5.1. [commutativity with dilaton factor] *Let (Y, g) be Lorentzian and φ is admissible to ∇ and Lorentzian. Then $e^{-\varphi^\diamond \Phi}$ and $\sqrt{-\text{SymDet}_X(\varphi^\diamond(g+B) + 2\pi\alpha'F_\nabla)}$ commute. Similarly, for the case (Y, g) Lorentzian and φ admissible and spacelike, and the case (Y, g) Riemannian and φ admissible and Riemannian.*

Proof. Since this is a pointwise issue over X , we only need to prove the following statement:

- Let $m_0, m_1 \in M_{r \times r}(\mathbb{C})$ commute. Assume that $\text{Det}(m_1) \neq 0$ and let $\sqrt{m_1}$ be any of its square root. Then, m_0 and $\sqrt{m_1}$ also commute.

Since m_0 and m_1 commute, there exists a decomposition

$$m_0 = a_0 + n_0 \quad m_1 = a_1 + n_1$$

such that a_0 and a_1 are diagonalizable, n_0 and n_1 are nilpotent, and a_0, a_1, n_0, n_1 commute with each other. Since $\sqrt{m_1}$ can be expressed as a polynomial in n_1 with coefficients the evaluation of smooth functions on a_1 , the statement follows. \square

Remark 3.1.5.2. [dilaton factor] In proving Lemma 3.1.5.1 above, the fact that F_∇ lies in the commutant of A_φ for φ admissible to ∇ is used. When this condition is not satisfied, one may have to consider taking the symmetrized product of $e^{-\varphi^\diamond \Phi}$ and $\sqrt{-\text{SymDet}_X(\varphi^\diamond(g+B) + 2\pi\alpha'F_\nabla)}$.

This resolves Issue (5) in the list.

3.1.6 Reality of the trace

Let $\varphi : (X^{A_z}, E; \nabla) \rightarrow (Y, g, B, \Phi)$ be an admissible Lorentzian map to a Lorentzian manifold with a B -field B and a dilaton field Φ . As noted in Remark 3.1.4.8, the $End_{\mathbb{C}}(E)$ -valued tensor $\sqrt{-SymDet_X(\varphi^\diamond g)}$ is indeed A_φ -valued, and, hence,

- The factor $e^{-\varphi^\diamond \Phi}$ and the principal square root $\sqrt{-SymDet_X(\varphi^\diamond g)}$ commute.

It follows then from the proof of Lemma 3.1.4.7, the positivity of eigenvalues of $e^{-\varphi^\diamond \Phi}$, and the commutivity of $e^{-\varphi^\diamond \Phi}$ and $\sqrt{-SymDet_X(\varphi^\diamond g)}$ that

- $Tr(e^{-\varphi^\diamond \Phi} \sqrt{-SymDet_X(\varphi^\diamond g)})$ is positive real-valued (when applied to a frame on X that is compatible with the orientation).

Recall from the near end of Sec. 3.1.4, as a consequence of Lemma 3.1.4.7, that if ∇ and B are such that F_∇ and $\varphi^\diamond B$ are small in the sense that they are close enough to the zero-section of $(T^*X)^{\otimes 2} \otimes_{\mathbb{R}} End_{\mathbb{C}}(E)$ (with respect to the natural topology on the total space thereof), then the principal square root

$$\sqrt{-SymDet_X(\varphi^\diamond(g+B) + 2\alpha' F_\nabla)} \in C^\infty(\wedge^m T^*X \otimes_{\mathbb{R}} End_{\mathbb{C}}(E))$$

exists. It follows that, as a deformation of $Tr(e^{-\varphi^\diamond \Phi} \sqrt{-SymDet_X(\varphi^\diamond g)})$,

$$Tr \left(e^{-\varphi^\diamond \Phi} \sqrt{-SymDet_X(\varphi^\diamond(g+B) + 2\alpha' F_\nabla)} \right) \in C^\infty(\wedge^m T^*X)^{\mathbb{C}},$$

has positive real part if B and F_∇ are small enough. However, it may not be real itself. This can be remedied by taking only the real part of the resulting trace as the action functional.

Similarly, for the case where (Y, g) is Lorentzian and (φ, ∇) is admissible spacelike or the case where (Y, g) is Riemannian and (φ, ∇) is admissible Riemannian.

This resolves Issue (6) in the list.

We have thus resolved all of Issues (2) – (6) in the list.

3.2 The Dirac-Born-Infeld action for admissibles pairs (φ, ∇)

With the preparations in Sec. 2.2 and Sec. 3.1, we can now define the Dirac-Born-Infeld action for D-branes along the line of [L-Y1] (D(1)) and [L-Y4] (D(11.1)).

Definition 3.2.1. [Dirac-Born-Infeld action for admissible (φ, ∇)] (1) Let (Y, Φ, g, B) be a Lorentzian manifold (Y, g) with a B -field B and a dilaton field Φ , and $\varphi : (X^{A_z}, E; \nabla) \rightarrow Y$ be an admissible Lorentzian map. We assume that B and the curvature F_∇ are small enough. Then the *Dirac-Born-Infeld action* $S_{DBI}^{(\Phi, g, B)}$ for the pair (φ, ∇) is defined to be

$$S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla) := -T_{m-1} \int_X Re \left(Tr \left(e^{-\varphi^\diamond \Phi} \sqrt{-SymDet_X(\varphi^\diamond(g+B) + 2\pi\alpha' F_\nabla)} \right) \right),$$

where $m = \dim X$, T_{m-1} is the D($m-1$)-brane tension, α' is the Regge slope, and $2\pi\alpha'$ is the inverse of the open-string tension.

(2) Let either (Y, g) Lorentzian and $\varphi : (X^{A_z}, E; \nabla) \rightarrow (Y, \Phi, g, B)$ admissible and spacelike or (Y, g) Riemannian and $\varphi : (X^{A_z}, E; \nabla) \rightarrow (Y, \Phi, g, B)$ admissible and Riemannian. Assume also that B and F_∇ are small enough. Then the *Dirac-Born-Infeld action* $S_{DBI}^{(\Phi, g, B)}$ for the pair (φ, ∇) is defined to be

$$S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla) := T_{m-1} \int_X Re \left(Tr \left(e^{-\varphi^\diamond \Phi} \sqrt{SymDet_X(\varphi^\diamond(g+B) + 2\pi\alpha' F_\nabla)} \right) \right).$$

In local coordinate chart $U \subset X$, $V \subset Y$, this is explicitly

$$S_{DBI}^{(\Phi, g, B)}(\varphi|_U, \nabla|_U) = \mp T_{m-1} \int_U \text{Tr} e^{-\varphi^\sharp(\Phi)} \sqrt{\mp \text{SymDet}_U \left(\sum_{i,j} \varphi^\sharp(g_{ij} + B_{ij}) D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) + 2\pi\alpha' F_{\mu\nu} \right)_{\mu\nu}} d^m x.$$

Theorem 3.2.2. [non-Abelian Dirac-Born-Infeld action for D-brane world-volume]

Under the assumption of the enough weakness of the B -field B on Y and the gauge curvature F_∇ on X , the Dirac-Born-Infeld action $S_{DBI}^{(\Phi, g, B)}$ for an admissible pair (φ, ∇) in each setting in Definition 3.2.1 is well-defined. Furthermore, when the rank r of E , as a complex vector bundle over X , is 1 (i.e. the case of a simple D-brane where the Chan-Paton bundle E is a complex line bundle), the action $S_{DBI}^{(\Phi, g, B)}$ as defined therein resumes to the standard Dirac-Born-Infeld action in the string-theory literature action for simple D-branes moving in a space-time with a background metric, B -field, and dilaton field, e.g., [Po3: vol. I, Eqn. (8.7.2)].

Proof. The discussions in Sec. 2.2 on admissible pairs (φ, ∇) and in Sec. 3.1.1 – Sec. 3.1.4 and the assumption that B and F_∇ are weak enough imply that the principal square root

$$\sqrt{\pm \text{SymDet}_X(\varphi^\circ(g + B) + 2\pi\alpha' F_\nabla)}$$

in each case is well-defined and has positive real part. This implies the well-definedness and $\mathbb{R}_{>0}$ -valuedness of the whole integrand in each case of the statement.

That $S_{DBI}^{(\Phi, g, B)}$ as defined is a generalization of the standard Dirac-Born-Infeld action in, for example, the quoted textbook by Polchinski, is immediate. □

Remark 3.2.3. [on the overall sign in $S_{DBI}^{(\Phi, g, B)}$] Recall that in electrodynamics, the action for a relativistic charged particle of mass m and electric charge e moving in a space-time (Y, g) with a background $U(1)$ -gauge field A' on Y , whose curvature gives the electromagnetic field on Y , is given by the Lorentz-invariant action

$$S_{EM}(\gamma) = -m \int_{\mathbb{R}^1} \sqrt{-g(\frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau})} d\tau + e \int_{\mathbb{R}^1} \gamma^* A'.$$

Here, $\gamma : \mathbb{R}^1 \rightarrow Y$ is the world-line of the particle, parameterized by τ . In comparison with the situation for D-branes, γ here corresponds to the admissible map φ , A' here plays the role of a Ramond-Ramond field C , the first term in S_{EM} corresponds to the Dirac-Born-Infeld action $S_{DBI}^{(g)}$, and the second term in S_{EM} corresponds to the Chern-Simons/Wess-Zumino action $S_{CS/WZ}^{(C)}$ (cf. Sec. 6). See, e.g., [Ja: Chapter 12]. This comparison sets the overall sign in $S_{DBI}^{(\Phi, g, B)}$ to be $-T_{m-1} \int_X(\cdots)$, rather than $+T_{m-1} \int_X(\cdots)$ for φ Lorentzian.

For φ spacelike or Riemannian, we set the sign to be $+$, by convention, to fit in with the study of minimal submanifolds and harmonics maps (e.g. [La], [L-W]).

Remark 3.2.4. [open-string-compatible quantizable action and super generalization]. For the first time since the beginning of this D-project, the dynamics of D-branes is addressed along the line of the project in a most natural and geometric way. This brings the study of D-branes truly to the same starting point as that for the fundamental string:

string theory	D-brane theory
<i>string world-sheet :</i> <i>2-manifold Σ</i>	<i>D-brane world-volume :</i> <i>Azumaya/matrix manifold</i> <i>with a fundamental module with a connection</i> <i>(X^{Az}, E, ∇)</i>
<i>string moving in space-time Y :</i> <i>differentiable map $f : \Sigma \rightarrow Y$</i>	<i>D-brane moving in space-time Y :</i> <i>differentiable map $\varphi : (X^{Az}, E, \nabla) \rightarrow Y$</i>
<i>Nambu-Goto action S_{NG} for f's</i>	<i>Dirac-Born-Infeld action S_{DBI} for (φ, ∇)'s</i>

On the other hand, from the lesson in string theory one learns that this action is quite unworkable for quantization of the theory. Thus, the above table should be extended immediately to the following not-yet-completed table as a guide for further studies:

string theory	D-brane theory
<i>string world-sheet :</i> <i>2-manifold Σ</i>	<i>D-brane world-volume :</i> <i>Azumaya/matrix manifold</i> <i>with a fundamental module with a connection</i> <i>(X^{Az}, E, ∇)</i>
<i>string moving in space-time Y :</i> <i>differentiable map $f : \Sigma \rightarrow Y$</i>	<i>D-brane moving in space-time Y :</i> <i>differentiable map $\varphi : (X^{Az}, E, \nabla) \rightarrow Y$</i>
<i>Nambu-Goto action S_{NG} for f's</i>	<i>Dirac-Born-Infeld action S_{DBI} for (φ, ∇)'s</i>
<i>Polyakov action $S_{Polyakov}$ for bosonic strings</i>	???, required to be open-string compatible
<i>action for Ramond-Neveu-Schwarz superstrings</i>	???, cf. [L-Y5: Sec. 5.1] (D(11.2))
<i>action for Green-Schwarz superstrings</i>	???, cf. [L-Y5: Sec. 5.1] (D(11.2))
<i>quantization</i>	???

Recall how the Dirac-Born-infeld action for a simple D-branes arises from the anomaly-free condition for the world-sheet of open-strings with end-points on such D-brane; cf. [Le]. Here, ‘open-string compatible’ means that the new quantizable action for D-branes is required to produce the same anomaly-free conditions for open strings.

Remark 3.2.5. [when in addition φ is Hermitian and ∇ unitary] With the setting in Definition 3.2.1, let E be equipped with a Hermitian structure \langle, \rangle . If in addition φ is Hermitian and ∇ is unitary with respect to \langle, \rangle , then one can check that

$$e^{-\varphi^\sharp(\Phi)} \sqrt{\mp \text{SymDet}_X(\varphi^\diamond(g+B)) + 2\pi\alpha' F_\nabla}$$

is Hermitian-matrix-valued (with respect to any local unitary frame on (E, \langle, \rangle)). and, hence, the Dirac-Born-Infeld action for an admissible Hermitian pair (φ, ∇) is simply

$$S_{DBI}(\varphi, \nabla) := \mp T_{m-1} \int_X e^{-\varphi^\diamond(\Phi)} \text{Tr} \sqrt{\mp \text{SymDet}_X(\varphi^\diamond(g+B)) + 2\pi\alpha' F_\nabla}.$$

Cf. Remark 2.3.8 and Remark 5.3.3.

4 Variations of φ^\sharp in terms of variations of local generators

Recall from [L-Y6: Sec. 2] (D(11.3.1)) that

- Let (y^1, \dots, y^n) be a coordinate system on \mathbb{R}^n and $\varphi^\sharp : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ be a ring-homomorphism over $\mathbb{R} \subset \mathbb{C}$. Then for any $f \in C^\infty(\mathbb{R}^n)$,

$$\varphi^\sharp(f) = f(\varphi^\sharp(y^1), \dots, \varphi^\sharp(y^n)).$$

Here for the Right Hand Side of the equality, $\varphi^\sharp(y^i) \in C^\infty(\text{End}_{\mathbb{C}}(E))$, for $i = 1, \dots, n$, and the value $f(\varphi^\sharp(y^1), \dots, \varphi^\sharp(y^n))$ is computed pointwise-over- X through the built-in/canonical C^∞ -ring structure of the commutative subalgebra generated by $\varphi^\sharp(y^1)(x), \dots, \varphi^\sharp(y^n)(x) \in \text{End}_{\mathbb{C}}(E|_x)$ for all $x \in X$. That the result lies in $C^\infty(\text{End}_{\mathbb{C}}(E))$ and coincides with $\varphi^\sharp(f)$ is proved using the Generalized Division Lemma as a consequence of the Malgrange Division Theorem, (cf. [L-Y6: Step (b) in Proof of Theorem 3.1.1] (D(11.3.1))). The equality says that

- A differentiable map $\varphi : (X^{\text{Az}}, E) \rightarrow \mathbb{R}^n$, defined by a ring-homomorphism $\varphi^\sharp : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$, is determined by the value of φ^\sharp on the coordinate functions y^1, \dots, y^n of \mathbb{R}^n ; namely by

$$\varphi^\sharp(y^1), \dots, \varphi^\sharp(y^n) \in C^\infty(\text{End}_{\mathbb{C}}(E)).$$

([L-Y6: Theorem 3.2.1] (D(11.3.1)).)

To calculate the variation of the Dirac-Born-Infeld S_{DBI} under variations of (φ, ∇) , one needs to address the following generalization of the above order-0 result to higher orders:

- Q.** Let $T = (-\varepsilon, \varepsilon)^l \subset \mathbb{R}^l$ be the base manifold and $\varphi_t : (X^{\text{Az}}, E) \rightarrow \mathbb{R}^n$, $\mathbf{t} := (t^1, \dots, t^l) \in T$, be a T -family of differentiable maps from (X^{Az}, E) to \mathbb{R}^n (with coordinate functions $\mathbf{y} := (y^1, \dots, y^n)$), defined by

$$\varphi_t^\sharp : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E)).$$

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_l)$, with $\alpha_1, \dots, \alpha_l \in \mathbb{Z}_{\geq 0}$, let $|\boldsymbol{\alpha}| := \alpha_1 + \dots + \alpha_l$ and

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{t}^{\boldsymbol{\alpha}}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial t^{1\alpha_1} \dots \partial t^{l\alpha_l}}.$$

Consider the derivation of order $|\boldsymbol{\alpha}| \geq 1$

$$\begin{aligned} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{t}^{\boldsymbol{\alpha}}} \varphi_t^\sharp &: C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E)) \\ f &\longmapsto \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{t}^{\boldsymbol{\alpha}}} (\varphi_t^\sharp(f)) \end{aligned}.$$

Then, for $f \in C^\infty(\mathbb{R}^n)$,

Can $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{t}^{\boldsymbol{\alpha}}} (\varphi_t^\sharp(f))$ be expressed in terms of $\left(\frac{\partial^{|\boldsymbol{\alpha}_1|}}{\partial \mathbf{t}^{\boldsymbol{\alpha}_1}} (\varphi_t^\sharp(y^1)), \dots, \frac{\partial^{|\boldsymbol{\alpha}_n|}}{\partial \mathbf{t}^{\boldsymbol{\alpha}_n}} (\varphi_t^\sharp(y^n)) \right)$'s with $\boldsymbol{\alpha}_1 + \dots + \boldsymbol{\alpha}_n = \boldsymbol{\alpha}$?

In this section, we answer this question affirmatively at the level of germs of differentiable functions, following a similar reasoning as in [L-Y6] (D(11.3.1)). The result will be used to calculate the first variation of $S_{DBI}^{(\Phi, g, B)}$ in the current notes and the second variation of $S_{DBI}^{(\Phi, g, B)}$ in a sequel.

4.1 $\varphi_t^\sharp(f)$ in terms of $(\varphi_t^\sharp(y^1), \dots, \varphi_t^\sharp(y^n))$ via Generalized Division Lemma

With the notations from above, we recall from [L-Y6] (D(11.3.1)) how the right-hand-side of the equality

$$\varphi_t^\sharp(f) = f(\varphi_t^\sharp(y^1), \dots, \varphi_t^\sharp(y^n))$$

is expressed in terms of f and $\varphi_t^\sharp(y^1), \dots, \varphi_t^\sharp(y^n)$ via the Generalized Division Lemma, a corollary of the Malgrange Division Theorem ([Mal]; see also [Brö], [Mat1], [Mat2], and [Ni]). Readers are referred to ibidem for more details.

The basic setup

For convenience, consider the projection map

$$X_T := X \times T \longrightarrow X$$

and denote the pull-back of E to $X \times T$ by E_T . Then the T -family of differentiable maps

$$\{\varphi_t : (X^{\text{Az}}, E) \rightarrow \mathbb{R}^n \mid t \in T\},$$

with φ_t defined by a ring-homomorphism $\varphi_t^\sharp : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ over $\mathbb{R} \subset \mathbb{C}$, defines a differentiable map

$$(X_T^{\text{Az}}, E_T) \xrightarrow{\varphi_T} \mathbb{R}^n$$

that is defined by the ring-homomorphism

$$C^\infty(\text{End}_{\mathbb{C}}(E_T)) \xleftarrow{\varphi_T^\sharp} C^\infty(\mathbb{R}^n)$$

over $\mathbb{R} \subset \mathbb{C}$ that restricts to φ_t^\sharp , for all $t \in T$. This extends canonically to a commutative diagram of ring-homomorphisms (over \mathbb{R} or $\mathbb{R} \subset \mathbb{C}$, whichever is applicable) ([L-Y6: Theorem 3.1.1] (D(11.3.1)))

$$\begin{array}{ccc} C^\infty(\text{End}_{\mathbb{C}}(E_T)) & \xleftarrow{\varphi_T^\sharp} & C^\infty(\mathbb{R}^n) \\ \uparrow & \searrow \tilde{\varphi}_T^\sharp & \downarrow pr_{\mathbb{R}^n}^\sharp \\ C^\infty(X_T) & \xhookrightarrow{pr_{X_T}^\sharp} & C^\infty(X_T \times \mathbb{R}^n), \end{array}$$

where $pr_{X_T} : X_T \times \mathbb{R}^n \rightarrow X_T$ and $pr_{\mathbb{R}^n} : X_T \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are the projection maps, and $C^\infty(X_T) \hookrightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ follows from the inclusion of the center $C^\infty(X_T)^{\mathbb{C}}$ of $C^\infty(\text{End}_{\mathbb{C}}(E_T))$. This in turn defines the following diagrams of differentiable maps that extends φ_T

$$\begin{array}{ccc} (X_T^{\text{Az}}, E_T) & \xrightarrow{\varphi_T} & \mathbb{R}^n \\ \downarrow & \searrow \tilde{\varphi}_T & \uparrow pr_{\mathbb{R}^n} \\ X_T & \xleftarrow{pr_{X_T}} & X_T \times \mathbb{R}^n. \end{array}$$

Let \mathcal{E}_T be the sheaf of C^∞ -sections of E_T . Then, the $\mathcal{O}_{X_T \times \mathbb{R}^n}^{\mathbb{C}}$ -module

$$\tilde{\mathcal{E}}_{\varphi_T} := \tilde{\varphi}_{T*}(\mathcal{E}_T)$$

defines the *graph* of φ_T . Its C^∞ -scheme-theoretical support

$$\Gamma_{\varphi_T} := \text{Supp}(\tilde{\mathcal{E}}_{\varphi_T}) \subset X_T \times \mathbb{R}^n$$

is finite and algebraic over X_T under the restriction of the projection $pr_{X_T} : X_T \times \mathbb{R}^n \rightarrow X_T$.

Γ_{φ_T} and the spectral subscheme Σ_{φ_T} of φ_T^\sharp in $X_T \times \mathbb{R}^n$

Under the projection map $pr_{\mathbb{R}^n} : X_T \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, the generators y^1, \dots, y^n of $C^\infty(\mathbb{R}^n)$, as a C^∞ -ring, pull back to elements in $C^\infty(X_T \times \mathbb{R}^n)$. They will still be denoted by y^1, \dots, y^n . Define the *spectral subscheme* of φ_T (associated to the generating set $\{y^1, \dots, y^n\}$ of $C^\infty(\mathbb{R}^n)$) to be the subscheme

$$\Sigma_{\varphi_T; \{y^1, \dots, y^n\}} \subset X_T \times \mathbb{R}^n$$

defined by the ideal

$$I_{\varphi_T; \{y^1, \dots, y^n\}} = (\det(y^i \cdot Id_{r \times r} - \varphi_T^\sharp(y^i)) \mid i = 1, \dots, n) \subset C^\infty(X_T \times \mathbb{R}^n).$$

Here, r is the rank of E as a complex vector bundle over X , and $Id_{r \times r} \in C^\infty(End_{\mathbb{C}}(E))$ the identity endomorphism. Then,

$$\Gamma_{\varphi_T} \subset \Sigma_{\varphi_T; \{y^1, \dots, y^n\}} \quad \text{with} \quad (\Gamma_{\varphi_T})_{\text{red}} = (\Sigma_{\varphi_T; \{y^1, \dots, y^n\}})_{\text{red}},$$

where $(\cdot)_{\text{red}}$ denotes the reduced subscheme of a scheme in the sense C^∞ -algebraic geometry. In terms of this picture, one has

- For any $f \in C^\infty(\mathbb{R}^n)$, denote its pull-back to $C^\infty(X_T \times \mathbb{R}^n)$ still by f . Then, $\varphi_T^\sharp(f)$ depends only the restriction of f to $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$. In other words, if $f_1, f_2 \in C^\infty(\mathbb{R}^n)$ satisfy that, after being pulled back to $X_T \times \mathbb{R}^n$, $f_1 - f_2 \in I_{\varphi_T; \{y^1, \dots, y^n\}}$, then $\varphi_T^\sharp(f_1) = \varphi_T^\sharp(f_2)$.

It is important to note that $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ is a C^∞ -subscheme, rather than just a subset, of $X_T \times \mathbb{R}^n$. Thus, the restriction of $f|_{\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}}$ of f on $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ captures not only the value of f at \mathbb{R} -points on $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ but also the behavior of f in an infinitesimal neighborhood of $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ up to a finite order. Cf. FIGURE 4-1-1.

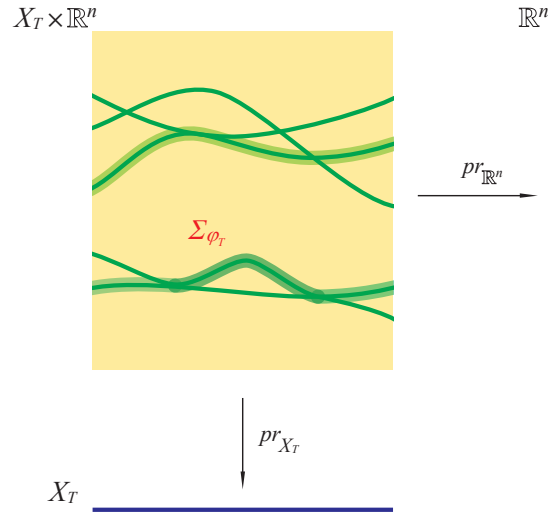


FIGURE 4-1-1. The spectral subscheme Σ_{φ_T} (in green color, with the green shade indicating the nilpotent structure/cloud on Σ_{φ_T}) in $X_T \times \mathbb{R}^n$ associated to a ring-homomorphism $\varphi_T^\sharp : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(End_{\mathbb{C}}(E_T))$. More than just a point-set with topology, it is a C^∞ -scheme that is finite over X_T .

Function germs at $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ from the Generalized Division Lemma à la Malgrange

Note that the polynomials $\det(y^i \cdot \text{Id}_{r \times r} - \varphi_T^\sharp(y^i)) \in C^\infty(X_T)[y^i] \subset C^\infty(X_T \times \mathbb{R}^n)$, $i = 1, \dots, n$, are of degree r . As a consequence of the Generalized Division Lemma, which is a corollary of the Malgrange Division Theorem, one has

- For $q \in \Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ and $h \in C^\infty(X_T \times \mathbb{R}^n)$, there exists a neighborhood $U'_h \times V_h$ of q in $X_T \times \mathbb{R}^n$ and a polynomial of (y^1, \dots, y^n) -degree $\leq (r-1, \dots, r-1)$

$$R_{h;q} := \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{h;q} \cdot (y^1)^{d_1} \cdots (y^n)^{d_n} \in C^\infty(U'_h)[y^1, \dots, y^n] \subset C^\infty(U'_h \times V_h)$$

such that

$$h|_{U'_h \times V_h} = R_{h;q} + h',$$

where $h' \in I_{\varphi_T; \{y^1, \dots, y^n\}}|_{U'_h \times V_h}$.

The explicit form for $\varphi_T^\sharp(f)$ from the Generalized Division Lemma

For $f \in C^\infty(\mathbb{R}^n)$ and $p \in X_T$, let

- $\{q_1, \dots, q_s\}$ be the set of \mathbb{R} -points in the 0-dimensional subscheme $pr_{X_T}^{-1}(p) \cap \Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ of $X_T \times \mathbb{R}^n$,

Then, for a neighborhood U' of $p \in X_T$ sufficiently small, $pr_{X_T}^{-1}(U') \cap \Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ has exactly s -many connected components

$$pr_{X_T}^{-1}(U') \cap \Sigma_{\varphi_T; \{y^1, \dots, y^n\}} = \Sigma_{(U'; 1)} \sqcup \cdots \sqcup \Sigma_{(U'; s)} \quad \text{with} \quad q_j \in \Sigma_{(U'; j)}.$$

This implies that the support $\text{Supp}(\tilde{\mathcal{E}}_{\varphi_T}|_{U' \times \mathbb{R}^n})$ of $\tilde{\mathcal{E}}_{\varphi_T}|_{U' \times \mathbb{R}^n}$ has also exactly s -many connected components. Let

- $E_T|_{U'} = E|_{U'}^{(1)} \oplus \cdots \oplus E|_{U'}^{(s)}$ be the decomposition of $E_T|_{U'}$ associated to the decomposition of $\tilde{\mathcal{E}}_{\varphi_T}|_{U' \times \mathbb{R}^n}$ into the direct sum of its restriction to the connected components of $\text{Supp}(\tilde{\mathcal{E}}_{\varphi_T}|_{U' \times \mathbb{R}^n})$.

In terms of this,

- Over U' , φ_T^\sharp is decomposed into

$$\varphi_T^\sharp = (\varphi_T^{\sharp, (1)}, \dots, \varphi_T^{\sharp, (s)}) = \varphi_T^{\sharp, (1)} \oplus \cdots \oplus \varphi_T^{\sharp, (s)},$$

with

$$\varphi_T^{\sharp, (j)} : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E_T|_{U'}^{(j)})),$$

for $j = 1, \dots, s$.

Now for each $\varphi_T^{\sharp, (j)} : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E_T|_{U'}^{(j)}))$, one can apply the result in the previous theme on the Generalized Division Lemma to express the germ of $\varphi_T^{\sharp, (j)}(f)$ over $p \in X_T$ in terms of $\varphi_T^{\sharp, (j)}(y^1), \dots, \varphi_T^{\sharp, (j)}(y^n)$ as follows.

- (1) First, it follows from the previous theme on the Generalized Division Lemma, with $q = q_j$ and $h = f$, that there exists a neighborhood $U_f'^{(j)} \times V_f^{(j)}$ of q_j in $U' \times \mathbb{R}^n$ and a polynomial of (y^1, \dots, y^n) -degree $\leq (r-1, \dots, r-1)$

$$R_{f;q_j} := \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{f;q_j} \cdot (y^1)^{d_1} \dots (y^n)^{d_n}$$

in $C^\infty(U_f'^{(j)})[y^1, \dots, y^n] \subset C^\infty(U_f'^{(j)} \times V_f^{(j)})$ such that

$$f|_{U_f'^{(j)} \times V_f^{(j)}} = R_{f;q_j} + f'_{(j)},$$

where $f'_{(j)} \in I_{\varphi_T; \{y^1, \dots, y^n\}}|_{U_f'^{(j)} \times V_f^{(j)}}$.

- (2) Then, over $U_f'^{(j)}$, $\varphi_T^{\sharp, (j)}(f)$ has the following expression in terms of $\varphi_T^{\sharp, (j)}(y^1), \dots, \varphi_T^{\sharp, (j)}(y^n)$:

$$\begin{aligned} \varphi_T^{\sharp, (j)}(f) &= \varphi_T^{\sharp, (j)}(R_{f;q_j}) \\ &= \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{f;q_j} \cdot (\varphi_T^{\sharp, (j)}(y^1))^{d_1} \dots (\varphi_T^{\sharp, (j)}(y^n))^{d_n} \\ &\in C^\infty(\text{End}_{\mathbb{C}}((E_T|_{U'}^{(j)})|_{U_f'^{(j)}})). \end{aligned}$$

- (3) Finally, let $U'_{f;p} := \bigcap_{j=1}^s U_f'^{(j)}$. Then, over $U'_{f;p}$,

$$\varphi_T^{\sharp}(f) = \varphi_T^{\sharp, (1)}(f) \oplus \dots \oplus \varphi_T^{\sharp, (s)}(f) \in C^\infty(\text{End}_{\mathbb{C}}(E_T|_{U'_{f;p}})).$$

Remark/Notation 4.1.1. [equivalent form: zero-th order] At the level of germs over X , one can re-write the above expression for $\varphi_T^{\sharp}(f)$ in terms of the evaluation of φ_T^{\sharp} on local coordinate functions more compactly as

$$\varphi_T^{\sharp}(f) = \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^f \cdot (\varphi_T^{\sharp}(y^1))^{d_1} \dots (\varphi_T^{\sharp}(y^n))^{d_n} \in C^\infty(\text{End}_{\mathbb{C}}(E_T|_{U'_f})),$$

where

$$a_{(d_1, \dots, d_n)}^f := \sum_{j=1}^s a_{(d_1, \dots, d_n)}^{f;q_j} \cdot \text{Id}_{(E_T|_{U'}^{(j)})|_{U_f'^{(j)}}}.$$

Here, for $j = 1, \dots, s$, we identify the identity map $\text{Id}_{(E_T|_{U'}^{(j)})|_{U_f'^{(j)}}}$ on $(E_T|_{U'}^{(j)})|_{U_f'^{(j)}}$ as an idempotent map on $E_T|_{U'_f}$ through its extension-by-zero on all $(E_T|_{U'}^{(j')})|_{U_f'^{(j')}$ for $j' \neq j$.

4.2 $\frac{\partial|\alpha|}{\partial t^\alpha}(\varphi_t^{\sharp}(f))$ in terms of $\left(\frac{\partial|\alpha_1|}{\partial t^{\alpha_1}}(\varphi_t^{\sharp}(y^1)), \dots, \frac{\partial|\alpha_n|}{\partial t^{\alpha_n}}(\varphi_t^{\sharp}(y^n))\right)$'s, $\alpha_1 + \dots + \alpha_n = \alpha$

With the preparation/review in Sec. 4.1, we now study how to compute/express $\frac{\partial|\alpha|}{\partial t^\alpha}(\varphi_t^{\sharp}(f))$ in terms of $\left(\frac{\partial|\alpha_1|}{\partial t^{\alpha_1}}(\varphi_t^{\sharp}(y^1)), \dots, \frac{\partial|\alpha_n|}{\partial t^{\alpha_n}}(\varphi_t^{\sharp}(y^n))\right)$'s with $\alpha_1 + \dots + \alpha_n = \alpha$.

4.2.1 Preparatory: Chain rule vs. Leibniz rule, and the increase of complexity

In the commutative case, this is simply the consequence of the chain rule of differentiations. However, in the noncommutative case, the formal commutative chain rule is not correct and there is no obvious chain rule to use. Nevertheless, the Leibniz rule

$$\partial.(ab) = (\partial.a)b + a\partial.b$$

still holds. Thus, for polynomial type functions with noncommutative arguments, one can still work things out. This is why the polynomial type expression (with coefficients in the germs of differentiable functions on X_T) of germs of $\varphi_T^\sharp(f)$ over X_T in terms of germs of $\varphi_T^\sharp(y^1), \dots, \varphi_T^\sharp(y^n)$ in the previous subsection is fundamental.

Example 4.2.1.1. [violation of formal chain rule in the noncommutative case] Consider $m(t) = a + bt$, where $ab \neq ba$ (but $ta = at$ and $tb = bt$). Let $f \in C^\infty(\mathbb{R}^1)$ defined by $f(y) = y^k$, $k \geq 2$. Then $\frac{d}{dy}f = ky^{k-1}$. The formal chain rule would give

$$\frac{d}{dt}(f(m(t))) = \left(\frac{d}{dy}f\right)(m(t)) \frac{d}{dt}m(t) = k(a+bt)^{k-1}b.$$

However, in truth,

$$\frac{d}{dt}(f(m(t))) = \frac{d}{dt}(\overbrace{(a+bt) \cdots (a+bt)}^{k\text{-times}}) = \sum_{k'=0}^{k-1} (a+bt)^l b (a+bt)^{k-1-k'}.$$

As b and $a+bt$ do not commute,

$$k(a+bt)^{k-1}b \neq \sum_{k'=0}^{k-1} (a+bt)^{k'} b (a+bt)^{k-1-k'}$$

in general.

Remark 4.2.1.2. [complexity] Example 4.2.1.1 serves to illustrate also the complexity of the expression in the noncommutative case versus the commutative case. In general, for $\alpha = (\alpha_1, \dots, \alpha_l)$ and a monomial

$$P(\xi^1, \dots, \xi^n) = (\xi^1)^{d_1} \cdots (\xi^n)^{d_n}$$

of degree $d = d_1 + \dots + d_n$ in noncommuting variables ξ^1, \dots, ξ^n , suppose that $\xi^i = \xi^i(\mathbf{t})$, where $\mathbf{t} = (t^1, \dots, t^l) \in T$ and assume that the order of differentiations play no role. Then, the direct expansion of the composition

$$\partial_{\mathbf{t}}^\alpha P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) := \frac{\partial^{|\alpha|}}{\partial \mathbf{t}^\alpha} P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t}))$$

via the Leibniz rule would have $d^{|\alpha|}$ -many terms. Which collapse to

$$\prod_{l'=1}^l \binom{\alpha_{l'} + d - 1}{d - 1} = \prod_{l'=1}^l \binom{\alpha_{l'} + d - 1}{\alpha_{l'}}$$

-many terms, after collecting like terms, of the form

$$\partial_{\mathbf{t}}^{\alpha_{(1)}} \xi^1(\mathbf{t}) \cdots \partial_{\mathbf{t}}^{\alpha_{(d_1)}} \xi^1(\mathbf{t}) \partial_{\mathbf{t}}^{\alpha_{(d_1+1)}} \xi^2(\mathbf{t}) \cdots \partial_{\mathbf{t}}^{\alpha_{(d_1+d_2)}} \xi^2(\mathbf{t}) \cdots \partial_{\mathbf{t}}^{\alpha_{(d_1+\dots+d_{n-1}+1)}} \xi^n(\mathbf{t}) \cdots \partial_{\mathbf{t}}^{\alpha_{(d)}} \xi^n(\mathbf{t}),$$

where $\alpha = \alpha_{(1)} + \dots + \alpha_{(d)}$, $\alpha_{(1)}, \dots, \alpha_{(d)} \in (\mathbb{Z}_{\geq 0})^l$, is an ordered partition $(\alpha_{(1)}, \dots, \alpha_{(d)})$ of α into d -many summands, with the coefficients from the coefficients of the expansion of the product $\prod_{l'=1}^l (z_{l'}^1 + \dots + z_{l'}^d)^{\alpha_{l'}}$.

Example 4.2.1.3. [differentiation via Leibniz rule] Let $\mathbf{t} = (t^1, t^2)$ and $a(\mathbf{t}), b(\mathbf{t})$ be functions of \mathbf{t} with values in a noncommutative ring. Then

$$\begin{aligned}
\partial_1^3 \partial_2^2 (a(\mathbf{t})^2 b(\mathbf{t})) &= \partial_1^3 \partial_2^2 a(\mathbf{t}) a(\mathbf{t}) b(\mathbf{t}) + \partial_2^2 a(\mathbf{t}) \partial_1^3 a(\mathbf{t}) b(\mathbf{t}) + \partial_2^2 a(\mathbf{t}) a(\mathbf{t}) \partial_1^3 b(\mathbf{t}) \\
&+ 3 \partial_1^2 \partial_2^2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) b(\mathbf{t}) + 3 \partial_1^2 \partial_2^2 a(\mathbf{t}) a(\mathbf{t}) \partial_1 b(\mathbf{t}) + 3 \partial_2^2 a(\mathbf{t}) \partial_1^2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) \\
&+ 3 \partial_1 \partial_2^2 a(\mathbf{t}) \partial_1^2 a(\mathbf{t}) b(\mathbf{t}) + 3 \partial_1 \partial_2^2 a(\mathbf{t}) a(\mathbf{t}) \partial_1^2 b(\mathbf{t}) + 3 \partial_2^2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_1^2 b(\mathbf{t}) \\
&+ 6 \partial_1 \partial_2^2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_1 b(\mathbf{t}) \\
&+ \partial_1^3 a(\mathbf{t}) \partial_2^2 a(\mathbf{t}) b(\mathbf{t}) + a(\mathbf{t}) \partial_1^3 \partial_2^2 a(\mathbf{t}) b(\mathbf{t}) + a(\mathbf{t}) \partial_2^2 a(\mathbf{t}) \partial_1^3 b(\mathbf{t}) \\
&+ 3 \partial_1^2 a(\mathbf{t}) \partial_1 \partial_2^2 a(\mathbf{t}) b(\mathbf{t}) + 3 \partial_1^2 a(\mathbf{t}) \partial_2^2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) + 3 a(\mathbf{t}) \partial_1^2 \partial_2^2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) \\
&+ 3 \partial_1 a(\mathbf{t}) \partial_1^2 \partial_2^2 a(\mathbf{t}) b(\mathbf{t}) + 3 \partial_1 a(\mathbf{t}) \partial_2^2 a(\mathbf{t}) \partial_1^2 b(\mathbf{t}) + 3 a(\mathbf{t}) \partial_1 \partial_2^2 a(\mathbf{t}) \partial_1^2 b(\mathbf{t}) \\
&+ 6 \partial_1 a(\mathbf{t}) \partial_1 \partial_2^2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) \\
&+ \partial_1^3 a(\mathbf{t}) a(\mathbf{t}) \partial_2^2 b(\mathbf{t}) + a(\mathbf{t}) \partial_1^3 a(\mathbf{t}) \partial_2^2 b(\mathbf{t}) + a(\mathbf{t})^2 \partial_1^3 \partial_2^2 b(\mathbf{t}) \\
&+ 3 \partial_1^2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_2^2 b(\mathbf{t}) + 3 \partial_1^2 a(\mathbf{t}) a(\mathbf{t}) \partial_1 \partial_2^2 b(\mathbf{t}) + 3 a(\mathbf{t}) \partial_1^2 a(\mathbf{t}) \partial_1 \partial_2^2 b(\mathbf{t}) \\
&+ 3 \partial_1 a(\mathbf{t}) \partial_1^2 a(\mathbf{t}) \partial_2^2 b(\mathbf{t}) + 3 \partial_1 a(\mathbf{t}) a(\mathbf{t}) \partial_1^2 \partial_2^2 b(\mathbf{t}) + 3 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_1^2 \partial_2^2 b(\mathbf{t}) \\
&+ 6 \partial_1 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_1 \partial_2^2 b(\mathbf{t}) \\
&+ 2 \partial_1^3 \partial_2 a(\mathbf{t}) \partial_2 a(\mathbf{t}) b(\mathbf{t}) + 2 \partial_2 a(\mathbf{t}) \partial_1^3 \partial_2 a(\mathbf{t}) b(\mathbf{t}) + 2 \partial_2 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_1^3 b(\mathbf{t}) \\
&+ 3 \cdot 2 \partial_1^2 \partial_2 a(\mathbf{t}) \partial_1 \partial_2 a(\mathbf{t}) b(\mathbf{t}) + 3 \cdot 2 \partial_1^2 \partial_2 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) + 3 \cdot 2 \partial_2 a(\mathbf{t}) \partial_1^2 \partial_2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) \\
&+ 3 \cdot 2 \partial_1 \partial_2 a(\mathbf{t}) \partial_1^2 \partial_2 a(\mathbf{t}) b(\mathbf{t}) + 3 \cdot 2 \partial_1 \partial_2 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_1^2 b(\mathbf{t}) + 3 \cdot 2 \partial_2 a(\mathbf{t}) \partial_1 \partial_2 a(\mathbf{t}) \partial_1^2 b(\mathbf{t}) \\
&+ 6 \cdot 2 \partial_1 \partial_2 a(\mathbf{t}) \partial_1 \partial_2 a(\mathbf{t}) \partial_1 b(\mathbf{t}) \\
&+ 2 \partial_1^3 \partial_2 a(\mathbf{t}) a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 2 \partial_2 a(\mathbf{t}) \partial_1^3 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 2 \partial_2 a(\mathbf{t}) a(\mathbf{t}) \partial_1^3 \partial_2 b(\mathbf{t}) \\
&+ 3 \cdot 2 \partial_1^2 \partial_2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 3 \cdot 2 \partial_1^2 \partial_2 a(\mathbf{t}) a(\mathbf{t}) \partial_1 \partial_2 b(\mathbf{t}) + 3 \cdot 2 \partial_2 a(\mathbf{t}) \partial_1^2 a(\mathbf{t}) \partial_1 \partial_2 b(\mathbf{t}) \\
&+ 3 \cdot 2 \partial_1 \partial_2 a(\mathbf{t}) \partial_1^2 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 3 \cdot 2 \partial_1 \partial_2 a(\mathbf{t}) a(\mathbf{t}) \partial_1^2 \partial_2 b(\mathbf{t}) + 3 \cdot 2 \partial_2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_1^2 \partial_2 b(\mathbf{t}) \\
&+ 6 \cdot 2 \partial_1 \partial_2 a(\mathbf{t}) \partial_1 a(\mathbf{t}) \partial_1 \partial_2 b(\mathbf{t}) \\
&+ 2 \partial_1^3 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 2 a(\mathbf{t}) \partial_1^3 \partial_2 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 2 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_1^3 \partial_2 b(\mathbf{t}) \\
&+ 3 \cdot 2 \partial_1^2 a(\mathbf{t}) \partial_1 \partial_2 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 3 \cdot 2 \partial_1^2 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_1 \partial_2 b(\mathbf{t}) + 3 \cdot 2 a(\mathbf{t}) \partial_1^2 \partial_2 a(\mathbf{t}) \partial_1 \partial_2 b(\mathbf{t}) \\
&+ 3 \cdot 2 \partial_1 a(\mathbf{t}) \partial_1^2 \partial_2 a(\mathbf{t}) \partial_2 b(\mathbf{t}) + 3 \cdot 2 \partial_1 a(\mathbf{t}) \partial_2 a(\mathbf{t}) \partial_1^2 \partial_2 b(\mathbf{t}) + 3 \cdot 2 a(\mathbf{t}) \partial_1 \partial_2 a(\mathbf{t}) \partial_1^2 \partial_2 b(\mathbf{t}) \\
&+ 6 \cdot 2 \partial_1 a(\mathbf{t}) \partial_1 \partial_2 a(\mathbf{t}) \partial_1 \partial_2 b(\mathbf{t})
\end{aligned}$$

which has $\binom{5}{2} \binom{4}{2} = 60$ -many terms (collapsed from an expansion of $3^5 = 243$ -many terms).

Notation 4.2.1.4. [differentiation via Leibniz rule on monomial] Continuing Remark 4.2.1.2 and Example 4.2.1.3. While the Leibniz rule itself is straightforward, the resulting expression after its repeated applications can be a burden. Some notations are introduced below for easier bookkeeping.

- (1) (*differentiation of any order*) Let $P\vec{tn}(\boldsymbol{\alpha}, d)$ be the set of ordered partitions of $\boldsymbol{\alpha}$ into d -many summands. For a $\vec{\pi} = (\boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(d)}) \in P\vec{tn}(\boldsymbol{\alpha}, d)$, denote

$$\begin{aligned}
[\partial_{\mathbf{t}}^{\vec{\pi}}] P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \\
:= \partial_{\mathbf{t}}^{\boldsymbol{\alpha}_{(1)}} \xi^1(\mathbf{t}) \dots \partial_{\mathbf{t}}^{\boldsymbol{\alpha}_{(d_1)}} \xi^1(\mathbf{t}) \partial_{\mathbf{t}}^{\boldsymbol{\alpha}_{(d_1+1)}} \xi^2(\mathbf{t}) \dots \partial_{\mathbf{t}}^{\boldsymbol{\alpha}_{(d_1+d_2)}} \xi^2(\mathbf{t}) \\
\dots \partial_{\mathbf{t}}^{\boldsymbol{\alpha}_{(d_1+\dots+d_{n-1}+1)}} \xi^n(\mathbf{t}) \dots \partial_{\mathbf{t}}^{\boldsymbol{\alpha}_{(d)}} \xi^n(\mathbf{t}).
\end{aligned}$$

Then, in terms of these,

$$\partial_{\mathbf{t}}^{\boldsymbol{\alpha}} P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) = \sum_{\vec{\pi} \in P\vec{tn}(\boldsymbol{\alpha}, d)} m_{\vec{\pi}} \cdot [\partial_{\mathbf{t}}^{\vec{\pi}}] P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})),$$

where $m_{\vec{\pi}} \in \mathbb{Z}_{>0}$ are the coefficients in the expansion.

- (2) (*first-order differentiation*) For the case $|\boldsymbol{\alpha}| = 1$, let $\partial_{\mathbf{t}}$ be the corresponding differentiable operator from the list $\partial_{t^1}, \dots, \partial_{t^l}$. Then, a $\vec{\pi} = (\boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(d)}) \in P\vec{tn}(\boldsymbol{\alpha}, d)$ has exactly one summand $\boldsymbol{\alpha}_{(j')}$ that is non-zero. Assume that $d_1 + \dots + d_{i-1} < j' \leq$

$d_1 + \cdots + d_{i-1} + d_i$ and let $1 \leq j := j' - (d_1 + \cdots + d_{i-1}) \leq d_i$. Then,

$$\begin{aligned} & [\partial_{\mathbf{t}}^{\vec{\pi}}]P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \\ &= \xi^1(\mathbf{t})^{d_1} \cdots \xi^{i-1}(\mathbf{t})^{d_{i-1}} \xi^i(\mathbf{t})^{j-1} \partial_t \xi^i(\mathbf{t}) \xi^i(\mathbf{t})^{d_i-j} \xi^{i+1}(\mathbf{t})^{d_{i+1}} \cdots \xi^n(\mathbf{t})^{d_n} \\ &=: ([\partial_{\xi^{i\vec{\pi}}}^{\vec{\pi}}]P)^L(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \cdot \partial_t \xi^{i\vec{\pi}}(\mathbf{t}) \cdot ([\partial_{\xi^{i\vec{\pi}}}^{\vec{\pi}}]P)^R(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})). \end{aligned}$$

In terms of these,

$$\begin{aligned} & \partial_t P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \\ &= \partial_{\mathbf{t}}^{\alpha} P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) = \sum_{\vec{\pi} \in \vec{Ptn}(\alpha, d)} [\partial_{\mathbf{t}}^{\vec{\pi}}]P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \\ &= \sum_{\vec{\pi} \in \vec{Ptn}(\alpha, d)} ([\partial_{\xi^{i\vec{\pi}}}^{\vec{\pi}}]P)^L(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \cdot \partial_t \xi^{i\vec{\pi}}(\mathbf{t}) \cdot ([\partial_{\xi^{i\vec{\pi}}}^{\vec{\pi}}]P)^R(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \\ & \hspace{15em} \text{(a summation of } d\text{-many terms)} \\ &= \sum_{i=1}^n \sum_{\vec{\pi} \in \vec{Ptn}(\alpha, d); i_{\vec{\pi}}=i} ([\partial_{\xi^i}^{\vec{\pi}}]P)^L(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \cdot \partial_t \xi^i(\mathbf{t}) \cdot ([\partial_{\xi^i}^{\vec{\pi}}]P)^R(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})) \\ & \hspace{15em} \text{(an expression closer to the usual chain rule).} \end{aligned}$$

Remark 4.2.1.5. [would-be chain rule in the noncommutative case] Readers are recommended to compare the last expression in Notation 4.2.1.4, Item (2), with the expression for the chain rule in the commutative case

$$\partial_t f(y^1(\mathbf{t}), \dots, y^n(\mathbf{t})) = \sum_{i=1}^n (\partial_{y^i} f)(y^1(\mathbf{t}), \dots, y^n(\mathbf{t})) \cdot \partial_t y^i(\mathbf{t}).$$

In a sense, noncommutativity brings into the problem the necessity to distinguish ‘*which* ξ^i *is involved*’ when we takes the differentiation $\partial_t(P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})))$. For that reason, each $\partial_{\xi^i} P(\xi^1, \dots, \xi^n)$ splits into two factors, the *left factor* $(\cdots)^L$ and the *right factor* $(\cdots)^R$ that depends on this additional detail.

Remark 4.2.1.6. [standard presentation for mixed case] For the situation to appear in the current notes, the values $\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t})$ for any \mathbf{t} commute among themselves but not necessarily with their differentiations with respect to \mathbf{t} . The above discussion still applies to such situations. However, the explicit expression for $\partial_{\mathbf{t}}^{\alpha} P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t}))$ after the expansion by Leibniz rule depends on how we represent $P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t}))$ in terms of a product of d_1 -many $\xi^1(\mathbf{t})$ ’s, \dots , d_n -many $\xi^n(\mathbf{t})$ ’s. By convention, we will take $\xi^1(\mathbf{t})^{d_1} \cdots \xi^n(\mathbf{t})^{d_n}$ as the standard presentation for $P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t}))$ and the resulting expansion the standard expression for $\partial_{\mathbf{t}}^{\alpha} P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t}))$ and $[\partial_{\mathbf{t}}^{\vec{\pi}}]P(\xi^1(\mathbf{t}), \dots, \xi^n(\mathbf{t}))$ ’s respectively.

Notation 4.2.1.7. [first-order differentiation via Leibniz rule on polynomial] Continuing Remark 4.2.1.2, Example 4.2.1.3, and Notation 4.2.1.4, Item (2); and recall α and the associated ∂_t . For a multi-degree $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}^n$, denote the monomial $(\xi^1)^{d_1} \cdots (\xi^n)^{d_n}$ by $\xi^{\mathbf{d}}$ or $P_{\mathbf{d}}(\xi)$ interchangeably, $\xi^1(\mathbf{t})^{d_1} \cdots \xi^n(\mathbf{t})^{d_n}$ by $\xi(\mathbf{t})^{\mathbf{d}}$ or $P_{\mathbf{d}}(\xi(\mathbf{t}))$ interchangeably, and the total degree $|\mathbf{d}| := d_1 + \cdots + d_n$. Let

$$P(\xi) := P(\xi^1, \dots, \xi^n) = \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} c_{\mathbf{d}} \xi^{\mathbf{d}} =: \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} c_{\mathbf{d}} P_{\mathbf{d}}(\xi)$$

be a polynomial in (ξ^1, \dots, ξ^n) with coefficients commutative with all of ξ^1, \dots, ξ^n . Then

$$\begin{aligned}
\partial_t P(\xi(t)) &= \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} c_{\mathbf{d}} \partial_t (\xi(t)^{\mathbf{d}}) \\
&= \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} c_{\mathbf{d}} \sum_{\vec{\pi} \in \vec{Pin}(\alpha, d)} ([\partial_{\xi^{i(\vec{\pi}, \mathbf{d})}}^{\vec{\pi}}] P_{\mathbf{d}})^L(\xi(t)) \cdot \partial_t \xi^{i(\vec{\pi}, \mathbf{d})}(t) \cdot ([\partial_{\xi^{i(\vec{\pi}, \mathbf{d})}}^{\vec{\pi}}] P_{\mathbf{d}})^R(\xi(t)) \\
&= \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} \sum_{\vec{\pi} \in \vec{Pin}(\alpha, d)} ([\partial_{\xi^{i(\vec{\pi}, \mathbf{d})}}^{\vec{\pi}}] P_{\mathbf{d}})^L(\xi(t)) \cdot \partial_t \xi^{i(\vec{\pi}, \mathbf{d})}(t) \cdot ([\partial_{\xi^{i(\vec{\pi}, \mathbf{d})}}^{\vec{\pi}}] P_{\mathbf{d}})^R(\xi(t)) \\
&= \sum_{i=1}^n \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} \sum_{\vec{\pi} \in \vec{Pin}(\alpha, d), i(\vec{\pi}, \mathbf{d})=i} ([\partial_{\xi^i}^{\vec{\pi}}] P_{\mathbf{d}})^L(\xi(t)) \cdot \partial_t \xi^i(t) \cdot ([\partial_{\xi^i}^{\vec{\pi}}] P_{\mathbf{d}})^R(\xi(t)).
\end{aligned}$$

Here, $P_{(\mathbf{d})} := c_{\mathbf{d}} P_{\mathbf{d}}$ is the multi-degree \mathbf{d} component of the polynomial P . This is the expression — a “virtual chain rule” in some sense — we will need for the current notes.

The main technical issue

Continuing now the setting and the study in Sec. 4.1, via the Leibniz rule one has at first an expansion to

$$\begin{aligned}
\left(\frac{\partial^{|\alpha|}}{\partial \mathbf{t}^{\alpha}} \varphi_T^{\sharp} \right) (f) &= \frac{\partial^{|\alpha|}}{\partial \mathbf{t}^{\alpha}} (\varphi_T^{\sharp}(f)) \\
&= \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \frac{\partial^{|\alpha|}}{\partial \mathbf{t}^{\alpha}} \left(a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\sharp, (j)}(y^1))^{d_1} \dots (\varphi_T^{\sharp, (j)}(y^n))^{d_n} \right)
\end{aligned}$$

over U'_f . Since the coefficients $a_{(d_1, \dots, d_n)}^{f; q_j}$ may depend also on \mathbf{t} , the expansion to the above summation via repeated applications of the Leibniz rule involves, in general, terms that depend on $\left(\frac{\partial^{|\alpha_1|}}{\partial \mathbf{t}^{\alpha_1}} (\varphi_T^{\sharp}(y^1)), \dots, \frac{\partial^{|\alpha_n|}}{\partial \mathbf{t}^{\alpha_n}} (\varphi_T^{\sharp}(y^n)) \right)$'s with $\alpha_1 + \dots + \alpha_n < \alpha$. The main technical issue is:

- *How to express the summation of such lower-order derivative terms in terms of $\left(\frac{\partial^{|\alpha_1|}}{\partial \mathbf{t}^{\alpha_1}} (\varphi_T^{\sharp}(y^1)), \dots, \frac{\partial^{|\alpha_n|}}{\partial \mathbf{t}^{\alpha_n}} (\varphi_T^{\sharp}(y^n)) \right)$'s with $\alpha_1 + \dots + \alpha_n = \alpha$ alone so that, in the end, $(\frac{\partial^{|\alpha|}}{\partial \mathbf{t}^{\alpha}} \varphi_T^{\sharp})(f)$ depends only on $\left(\frac{\partial^{|\alpha_1|}}{\partial \mathbf{t}^{\alpha_1}} (\varphi_T^{\sharp}(y^1)), \dots, \frac{\partial^{|\alpha_n|}}{\partial \mathbf{t}^{\alpha_n}} (\varphi_T^{\sharp}(y^n)) \right)$'s with $\alpha_1 + \dots + \alpha_n = \alpha$.*

4.2.2 The case of first-order derivations

Proposition 4.2.2.1. [first order derivation] *Denote by ∂_{\bullet} any of $\partial/\partial t^1, \dots, \partial/\partial t^l$. Let $f \in C^{\infty}(Y)$, regarded also as an element in $C^{\infty}(X_T \times Y)$ through the inclusion $pr_Y^{\sharp} : C^{\infty}(Y) \rightarrow C^{\infty}(X_T \times Y)$ whenever necessary. Then $(\partial_{\bullet} \varphi_T^{\sharp})(f) (= \partial_{\bullet}(\varphi_T^{\sharp}(f)))$ depends on $((\partial_{\bullet} \varphi_T^{\sharp})(y^1), \dots, (\partial_{\bullet} \varphi_T^{\sharp})(y^n)) (= (\partial_{\bullet}(\varphi_T^{\sharp}(y^1)), \dots, \partial_{\bullet}(\varphi_T^{\sharp}(y^n))))$. More explicitly, recall the defining equations*

$$\chi_{\varphi_T}^{(i)} := \det(y^i \cdot Id_{r \times r} - \varphi_T^{\sharp}(y^i)) = (y^i)^r + a_{r-1}^{(i)}(y^i)^{r-1} + \dots + a_1^{(i)} y^i + a_0^{(i)},$$

$i = 1, \dots, n$, for the spectral locus $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}} \subset X_T \times Y$ of φ_T over U'_f and that, around $q_j \in \Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$, there exist $a_{(d_1, \dots, d_n)}^{f; q_j} \in C^\infty(U'_f)$ and $Q_{(i)}^{f; q_j} \in C^\infty(U'_f \times V_f^{(j)})$, $0 \leq d_i \leq r-1$, $i = 1, \dots, n$, such that

$$f = \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (y^1)^{d_1} \dots (y^n)^{d_n} + \sum_{i=1}^n Q_{(i)}^{f; q_j} \chi_{\varphi_T}^{(i)}.$$

Then, in terms of these data encoded in φ_T and f , one has

$$\begin{aligned} (\partial \cdot \varphi_T^\#)(f) &= \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{f; q_j} \cdot \partial \cdot \left((\varphi_T^{\#, (j)}(y^1))^{d_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d_n} \right) \\ &\quad + \sum_{j=1}^s \sum_{i=1}^n \tilde{\varphi}_T(Q_{(i)}^{f; q_j}) \sum_{d=1}^r a_d^{(i)} \cdot \partial \cdot \left((\varphi_T^{\#, (j)}(y^i))^d \right), \end{aligned}$$

from which one can use the Leibniz rule to further express $\partial \cdot ((\varphi_T^{\#, (j)}(y^1))^{d_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d_n})$'s and $\partial \cdot ((\varphi_T^{\#, (j)}(y^i))^d)$'s into the desired form. (Here, $a_r^{(i)} = 1$ by convention, for $i = 1, \dots, n$.)

Proof. We proceed in three steps.

Step (1) : Identifying the seemingly problematic terms Over U'_f , one has

$$\begin{aligned} (\partial \cdot \varphi_T^\#)(f) &= \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \partial \cdot \left(a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\#, (j)}(y^1))^{d_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d_n} \right) \\ &= \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \partial \cdot a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\#, (j)}(y^1))^{d_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d_n} \\ &\quad + \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \sum_{d'_1=0}^{d_1-1} a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\#, (j)}(y^1))^{d'_1} \cdot \partial \cdot \varphi_T^{\#, (j)}(y^1) \cdot (\varphi_T^{\#, (j)}(y^1))^{d_1-1-d'_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d_n} \\ &\quad + \dots \\ &\quad + \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \sum_{d'_n=0}^{d_n-1} a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\#, (j)}(y^1))^{d_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d'_n} \cdot \partial \cdot \varphi_T^{\#, (j)}(y^n) \cdot (\varphi_T^{\#, (j)}(y^n))^{d_n-1-d'_n}. \end{aligned}$$

Terms that are not manifestly multilinear in $(\partial \cdot (\varphi_T^{\#, (j)}(y^1)), \dots, \partial \cdot (\varphi_T^{\#, (j)}(y^n)))$ lie in the first cluster $\sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \partial \cdot a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\#, (j)}(y^1))^{d_1} \dots (\varphi_T^{\#, (j)}(y^n))^{d_n}$.

Since terms with different j 's are independent from each other and, hence, can be treated separately, to avoid the burden of notation and without loss of generality, we assume from now on in the proof that $s = 1$ and drop the j label altogether.

Step (2) : Defining equations of the spectral locus come to play Recall the generators

$$\begin{aligned} \chi_{\varphi_T}^{(i)} &:= \det(y^i \cdot Id_{r \times r} - \varphi_T^\#(y^i)) \\ &=: (y^i)^r + a_{r-1}^{(i)}(y^i)^{r-1} + \dots + a_1^{(i)} y^i + a_0^{(i)} \\ &\in C^\infty(U'_f)[y^i] \subset C^\infty(U'_f \times V_f), \end{aligned}$$

for $i = 1, \dots, n$, of the ideal $I_{\varphi_T; \{y^1, \dots, y^n\}}|_{U'_f \times V_f}$ that defines the spectral locus $\Sigma_{\varphi_T; \{y^1, \dots, y^n\}}$ of φ_T over U'_f . By construction, there exist $Q_{(1)}^{f;q}, \dots, Q_{(n)}^{f;q} \in C^\infty(U'_f \times V_f)$ such that

$$f = \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{f;q} \cdot (y^1)^{d_1} \cdots (y^n)^{d_n} + \sum_{i'=1}^n Q_{(i')}^{f;q} \chi_{\varphi_T}^{(i')}$$

in $C^\infty(U'_f \times V_f)$. Since $f \in C^\infty(V_f)$ and, hence, has no dependence on \mathbf{t} , applying ∂_\bullet to both sides of the above identity gives

$$0 = \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \partial_\bullet a_{(d_1, \dots, d_n)}^{f;q} \cdot (y^1)^{d_1} \cdots (y^n)^{d_n} + \sum_{i'=1}^n \partial_\bullet Q_{(i')}^{f;q} \cdot \chi_{\varphi_T}^{(i')} + \sum_{i''=1}^n Q_{(i'')}^{f;q} \cdot \partial_\bullet \chi_{\varphi_T}^{(i'')}.$$

Applying $\tilde{\varphi}_T^\sharp$ to this identity and noticing that $\tilde{\varphi}_T^\sharp(\chi_{\varphi_T}^{(i')}) = 0$ for $i' = 1, \dots, n$, one now has

$$\sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} \partial_\bullet a_{(d_1, \dots, d_n)}^{f;q} \cdot (\varphi_T^\sharp(y^1))^{d_1} \cdots (\varphi_T^\sharp(y^n))^{d_n} = - \sum_{i''=1}^n \tilde{\varphi}_T^\sharp(Q_{(i'')}^{f;q}) \cdot \tilde{\varphi}_T^\sharp(\partial_\bullet \chi_{\varphi_T}^{(i'')}).$$

Step (3): Understanding $\tilde{\varphi}_T^\sharp(\partial_\bullet \chi_{\varphi_T}^{(i'')})$ Explicitly, one has

$$\partial_\bullet \chi_{\varphi_T}^{(i'')} = \partial_\bullet a_{r-1}^{(i'')} \cdot (y^{i''})^{r-1} + \cdots + \partial_\bullet a_1^{(i'')} \cdot y^{i''} + \partial_\bullet a_0^{(i'')}$$

and, hence,

$$\tilde{\varphi}_T^\sharp(\partial_\bullet \chi_{\varphi_T}^{(i'')}) = \partial_\bullet a_{r-1}^{(i'')} \cdot (\varphi_T^\sharp(y^{i''}))^{r-1} + \cdots + \partial_\bullet a_1^{(i'')} \cdot \varphi_T^\sharp(y^{i''}) + \partial_\bullet a_0^{(i'')},$$

for $i'' = 1, \dots, n$. On the other hand,

$$\tilde{\varphi}_T^\sharp(\chi_{\varphi_T}^{(i'')}) = (\varphi_T^\sharp(y^{i''}))^r + a_{r-1}^{(i'')} \cdot (\varphi_T^\sharp(y^{i''}))^{r-1} + \cdots + a_1^{(i'')} \cdot \varphi_T^\sharp(y^{i''}) + a_0^{(i'')} = 0$$

and, hence,

$$\partial_\bullet (\tilde{\varphi}_T^\sharp(\chi_{\varphi_T}^{(i'')})) = 0.$$

Which gives

$$\begin{aligned} \tilde{\varphi}_T^\sharp(\partial_\bullet \chi_{\varphi_T}^{(i'')}) &= - \left[\partial_\bullet (\varphi_T^\sharp(y^{i''}))^r + a_{r-1}^{(i'')} \cdot \partial_\bullet (\varphi_T^\sharp(y^{i''}))^{r-1} + \cdots + a_1^{(i'')} \cdot \partial_\bullet \varphi_T^\sharp(y^{i''}) \right] \\ &= - \left[\sum_{k_r=0}^{r-1} (\varphi_T^\sharp(y^{i''}))^{k_r} \cdot \partial_\bullet \varphi_T^\sharp(y^{i''}) \cdot (\varphi_T^\sharp(y^{i''}))^{r-1-k_r} \right. \\ &\quad + a_{r-1}^{(i'')} \cdot \sum_{k_{r-1}=0}^{r-2} (\varphi_T^\sharp(y^{i''}))^{k_{r-1}} \cdot \partial_\bullet \varphi_T^\sharp(y^{i''}) \cdot (\varphi_T^\sharp(y^{i''}))^{r-2-k_{r-1}} + \cdots \\ &\quad \left. + a_2^{(i'')} \cdot \left(\partial_\bullet \varphi_T^\sharp(y^{i''}) \cdot \varphi_T^\sharp(y^{i''}) + \varphi_T^\sharp(y^{i''}) \cdot \partial_\bullet \varphi_T^\sharp(y^{i''}) \right) + a_1^{(i'')} \cdot \partial_\bullet \varphi_T^\sharp(y^{i''}) \right]. \end{aligned}$$

This concludes the proof of the proposition. \square

4.2.3 Generalization to derivations of any order

Thinking deep enough of the case of first order derivations leads one to its generalization to all higher-order situations. To state the proposition, with the notation from the previous theme, note that by shrinking U'_f if necessary and applying the Generalized Division Lemma repeatedly, first to f , then to $Q_{(i)}^{f;q_j}$, \dots , and so on, one has

$$\begin{aligned} f &= R_0^{f;q_j} + \sum_{i=1}^n R_{(i)}^{f;q_j} \chi_{\varphi_T}^{(i)} + \sum_{i_1, i_2=1}^n R_{(i_1, i_2)}^{f;q_j} \chi_{\varphi_T}^{(i_1)} \chi_{\varphi_T}^{(i_2)} + \dots \\ &\quad + \sum_{i_1, \dots, i_k=1}^n R_{(i_1, \dots, i_k)}^{f;q_j} \chi_{\varphi_T}^{(i_1)} \dots \chi_{\varphi_T}^{(i_k)} + \sum_{i_1, \dots, i_{k+1}=1}^n Q_{(i_1, \dots, i_{k+1})}^{f;q_j} \chi_{\varphi_T}^{(i_1)} \dots \chi_{\varphi_T}^{(i_{k+1})} \\ &=: R^{f;q_j}[k] + \sum_{i_1, \dots, i_{k+1}=1}^n Q_{(i_1, \dots, i_{k+1})}^{f;q_j} \chi_{\varphi_T}^{(i_1)} \dots \chi_{\varphi_T}^{(i_{k+1})} \end{aligned}$$

around each q_j , for any $k \in \mathbb{Z}_{\geq 1}$. Here, $R_0^{f;q_j}$, $R_{(i)}^{f;q_j}$, \dots , $R_{(i_1, \dots, i_k)}^{f;q_j}$, and, hence, $R^{f;q_j}[k]$ are all in $C^\infty(U'_f)[y^1, \dots, y^n] \subset C^\infty(U'_f \times V_f^{(j)})$. Denote the multi-degree of a summand of such a polynomial by $\mathbf{d} = (d_1, \dots, d_n)$ and $(y^1)^{d_1} \dots (y^n)^{d_n}$ by $\mathbf{y}^{\mathbf{d}}$.

Proposition 4.2.3.1. [derivation of any order] *Let ∂^α be a derivation of order $k \geq 1$ with respect to the coordinates $\mathbf{t} = (t^1, \dots, t^l)$ of T . Then, over U'_f ,*

$$(\partial^\alpha \varphi_T^\#)(f) := \partial^\alpha(\varphi_T^\#(f)) = \sum_{j=1}^s R^{f;q_j}[k] \Big|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial^\alpha(\varphi_T^\#(j)(\mathbf{y}^{\mathbf{d}}))}$$

and, hence, depends only on $\left(\frac{\partial^{|\alpha_1|}}{\partial \mathbf{t}^{\alpha_1}}(\varphi_T^\#(j)(y^1)), \dots, \frac{\partial^{|\alpha_n|}}{\partial \mathbf{t}^{\alpha_n}}(\varphi_T^\#(j)(y^n)) \right)$'s, $j = 1, \dots, s$, with $\alpha_1 + \dots + \alpha_n = \alpha$. Here, $\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial^\alpha(\varphi_T^\#(j)(\mathbf{y}^{\mathbf{d}}))$ means “the replacement of $\mathbf{y}^{\mathbf{d}}$ by $\partial^\alpha(\varphi_T^\#(j)(\mathbf{y}^{\mathbf{d}}))$ for all multi-degree- \mathbf{d} summands of $R^{f;q_j}[k]$, \mathbf{d} running from $(0, \dots, 0)$ to $((k+1)r-1, \dots, (k+1)r-1)$ ”.

Proof. Since $|\alpha| = k$, there is always at least one $\chi_{\varphi_T}^{(\bullet)}$ factor left in the summands of the final expansion of $\partial^\alpha(Q_{(i_1, \dots, i_{k+1})}^{f;q_j} \chi_{\varphi_T}^{(i_1)} \dots \chi_{\varphi_T}^{(i_{k+1})})$ through repeating the Leibniz rule. Together with the identity $\tilde{\varphi}_T^\#(\chi_{\varphi_T}^{(\bullet)}) = 0$, one has

$$(\partial^\alpha \varphi_T^\#)(f) = \sum_{j=1}^s \partial^\alpha(R^{f;q_j}[k]).$$

By construction, for each j , $R^{f;q_j}[k]$ is in $C^\infty(U'_f)[y^1, \dots, y^n] \subset C^\infty(U'_f \times V_f^{(j)})$ of (y^1, \dots, y^n) -multi-degree $\leq ((k+1)r-1, \dots, (k+1)r-1)$. For convenience and all we need in the proof, we will write $R^{f;q_j}[k]$ as a polynomial in $\mathbf{y} = (y^1, \dots, y^n)$

$$R^{f;q_j}[k] = \sum_{\mathbf{d}} c_{j,\mathbf{d}} \mathbf{y}^{\mathbf{d}}$$

with the coefficients $c_{j,\mathbf{d}} \in C^\infty(U'_f)$. Recall from Sec. 4.1 the decomposition

$\varphi_T^\# = (\varphi_T^{\#(1)}, \dots, \varphi_T^{\#(s)})$, which induces the decomposition $\tilde{\varphi}_T^\# = (\tilde{\varphi}_T^{\#(1)}, \dots, \tilde{\varphi}_T^{\#(s)})$. Since $f \in C^\infty(Y)$,

$$\partial \cdot f = 0$$

for ∂_\bullet any of $\partial/\partial t^1, \dots, \partial/\partial t^l$. This gives a collection of identities for each j :

$$\begin{aligned}
0 &= \tilde{\varphi}_T^{\sharp, (j)}(\partial^\alpha f) = \sum_d \partial^\alpha c_{j,d} \cdot \varphi_T^{\sharp, (j)}(\mathbf{y}^d), \\
0 &= \partial^{\alpha_1}(\tilde{\varphi}_T^{\sharp, (j)}(\partial^{\alpha-\alpha_1} f)) = \tilde{\varphi}_T^{\sharp, (j)}(\partial^\alpha f) + \sum_d \partial^{\alpha-\alpha_1} c_{j,d} \cdot \partial^{\alpha_1} \varphi_T^{\sharp, (j)}(\mathbf{y}^d), \\
0 &= \partial^{\alpha_2}(\tilde{\varphi}_T^{\sharp, (j)}(\partial^{\alpha-\alpha_2} f)) \\
&= \tilde{\varphi}_T^{\sharp, (j)}(\partial^\alpha f) + \sum_{\substack{\alpha' \prec \alpha_2, \\ |\alpha'|=1}} m_{\alpha' \prec \alpha_2} \cdot \sum_d \partial^{\alpha-\alpha'} c_{j,d} \cdot \partial^{\alpha'} \varphi_T^{\sharp, (j)}(\mathbf{y}^d) + \sum_d \partial^{\alpha-\alpha_2} c_{j,d} \cdot \partial^{\alpha_2} \varphi_T^{\sharp, (j)}(\mathbf{y}^d), \\
0 &= \partial^{\alpha_3}(\tilde{\varphi}_T^{\sharp, (j)}(\partial^{\alpha-\alpha_3} f)) \\
&= \tilde{\varphi}_T^{\sharp, (j)}(\partial^\alpha f) + \sum_{\substack{\alpha' \prec \alpha_3, \\ |\alpha'|=1}} m_{\alpha' \prec \alpha_3} \cdot \sum_d \partial^{\alpha-\alpha'} c_{j,d} \cdot \partial^{\alpha'} \varphi_T^{\sharp, (j)}(\mathbf{y}^d) \\
&\quad + \sum_{\substack{\alpha'' \prec \alpha_3, \\ |\alpha''|=2}} m_{\alpha'' \prec \alpha_3} \cdot \sum_d \partial^{\alpha-\alpha''} c_{j,d} \cdot \partial^{\alpha''} \varphi_T^{\sharp, (j)}(\mathbf{y}^d) + \sum_d \partial^{\alpha-\alpha_3} c_{j,d} \cdot \partial^{\alpha_3} \varphi_T^{\sharp, (j)}(\mathbf{y}^d), \\
&\dots\dots\dots \\
0 &= \partial^{\alpha_{k'}}(\tilde{\varphi}_T^{\sharp, (j)}(\partial^{\alpha-\alpha_{k'}} f)) \\
&= \sum_{k''=0}^{k'-1} \sum_{\substack{\alpha'' \prec \alpha_{k'}, \\ |\alpha''|=k''}} m_{\alpha'' \prec \alpha_{k'}} \cdot \sum_d \partial^{\alpha-\alpha''} c_{j,d} \cdot \partial^{\alpha''} \varphi_T^{\sharp, (j)}(\mathbf{y}^d) + \sum_d \partial^{\alpha-\alpha_{k'}} c_{j,d} \cdot \partial^{\alpha_{k'}} \varphi_T^{\sharp, (j)}(\mathbf{y}^d), \\
&\dots\dots\dots
\end{aligned}$$

where, with a slight abuse of the labelling index k' , $\alpha_{k'}$ runs over all $\alpha_{k'} \prec \alpha$ with $|\alpha_{k'}| = k'$, $k' = 1, \dots, |\alpha| - 1$. Here,

$$m_{\alpha'' \prec \alpha'} = \binom{\alpha'_1}{\alpha''_1} \cdots \binom{\alpha'_l}{\alpha''_l}$$

counts the number of ways to choose $\partial^{\alpha''}$ from $\partial^{\alpha'}$ for $\alpha'' := (\alpha''_1, \dots, \alpha''_l) \prec \alpha' := (\alpha'_1, \dots, \alpha'_l)$.

Now observe that the above system of identities is equivalent to the following system of identities for each j :

$$\sum_d \partial^{\alpha-\alpha''} c_{j,d} \cdot \partial^{\alpha''} \varphi_T^{\sharp, (j)}(\mathbf{y}^d) = 0 \quad \text{for all } \alpha'' \prec \alpha, \ 0 \leq |\alpha''| < |\alpha| - 1.$$

It follows that

$$\begin{aligned}
\partial^\alpha(\varphi_T^\sharp(f)) &= \sum_{j=1}^s \sum_{k'=0}^{|\alpha|-1} \sum_{\substack{\alpha' \prec \alpha, \\ |\alpha'|=k'}} m_{\alpha' \prec \alpha} \cdot \sum_d \partial^{\alpha-\alpha'} c_{j,d} \cdot \partial^{\alpha'} \varphi_T^{\sharp, (j)}(\mathbf{y}^d) + \sum_{j=1}^s \sum_d c_{j,d} \cdot \partial^\alpha \varphi_T^{\sharp, (j)}(\mathbf{y}^d) \\
&= \sum_{j=1}^s \sum_d c_d \cdot \partial^\alpha \varphi_T^{\sharp, (j)}(\mathbf{y}^d) = \sum_{j=1}^s R^{f; q_j}[k] \big|_{\mathbf{y}^d \rightsquigarrow \partial^\alpha(\varphi_T^{\sharp, (j)}(\mathbf{y}^d))}.
\end{aligned}$$

This concludes the proof of the proposition. \square

Remark 4.2.3.2. [case $k = 1$] Since $Q_{(i_1, \dots, i_k)}^{f; q_j} = R_{(i_1, \dots, i_k)}^{f; q_j} + \sum_{i_{k+1}=1}^n Q_{(i_1, \dots, i_k, i_{k+1})}^{f; q_j} \chi_{\varphi_T}^{(i_{k+1})}$ and $\tilde{\varphi}_T^{\sharp, (j)}(\chi_{\varphi_T}^{(i_{k+1})}) = 0$ for $i_{k+1} = 1, \dots, n$, one has

$$\tilde{\varphi}_T^{\sharp, (j)}(Q_{(i_1, \dots, i_k)}^{f; q_j}) = \tilde{\varphi}_T^{\sharp, (j)}(R_{(i_1, \dots, i_k)}^{f; q_j}).$$

In particular, for $k = 1$,

$$\begin{aligned} & (R_{(i)}^{f; q_j} \chi_{\varphi_T}^{(i)})|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial_{\bullet} \varphi_T^{\sharp, (j)}(\mathbf{y}^{\mathbf{d}})} \\ &= (R_{(i)}^{f; q_j}|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial_{\bullet} \varphi_T^{\sharp, (j)}(\mathbf{y}^{\mathbf{d}})}) \cdot \tilde{\varphi}_T^{\sharp, (j)}(\chi_{\varphi_T}^{(i)}) + \tilde{\varphi}_T^{\sharp, (j)}(R_{(i)}^{f; q_j}) \cdot (\chi_{\varphi_T}^{(i)}|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial_{\bullet} \varphi_T^{\sharp, (j)}(\mathbf{y}^{\mathbf{d}})}) \\ &= \tilde{\varphi}_T^{\sharp, (j)}(Q_{(i)}^{f; q_j}) \cdot (\chi_{\varphi_T}^{(i)}|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial_{\bullet} \varphi_T^{\sharp, (j)}(\mathbf{y}^{\mathbf{d}})}) \end{aligned}$$

and Proposition 4.2.3.1 resumes to Proposition 4.2.2.1.

Remark 4.2.3.3. [case $k = 0$] Setting the convention that for $|\alpha| = 0$, i.e. $\alpha = (0, \dots, 0)$, $\partial^\alpha(\dots) := (\dots)$. Then, for $|\alpha| = 0$, Proposition 4.2.3.1 resumes to the case studied in [L-Y6] (D(11.3.1)), reviewed in Sec. 4.1; cf. the formula

$$\varphi_T^\sharp(f) = \sum_{j=1}^s \sum_{\substack{0 \leq d_i \leq r-1 \\ 1 \leq i \leq n}} a_{(d_1, \dots, d_n)}^{f; q_j} \cdot (\varphi_T^{\sharp, (j)}(y^1))^{d_1} \dots (\varphi_T^{\sharp, (j)}(y^n))^{d_n}$$

at the end of Sec. 4.1, which is simply $\sum_{j=1}^s R^{f; q_j}[0]|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \varphi_T^{\sharp, (j)}(\mathbf{y}^{\mathbf{d}})}$.

Remark 4.2.3.4. [A second look at Proposition 4.2.3.1 from a comparison with the commutative case] In the commutative case, let $X = \mathbb{R}^m$ with coordinates $\mathbf{x} = (x^1, \dots, x^m)$, $Y = \mathbb{R}^n$ with coordinates $\mathbf{y} = (y^1, \dots, y^n)$, $f \in C^\infty(Y)$, and $h := (h^1, \dots, h^n) : X \rightarrow Y$ be a differentiable map. Let T be a small neighborhood of the origin $0 \in \mathbb{R}^l$ with coordinates $\mathbf{t} = (t^1, \dots, t^l)$ and $h_T := (h_T^1, \dots, h_T^n) : X \rightarrow Y$ be a T -family of differentiable maps from X to Y that extends $h := h_0$. For $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{Z}_{\geq 0}^l$, let $\partial_{\mathbf{t}, 0}^\alpha := \partial^{|\alpha|} / (\partial t^1)^{\alpha_1} \dots (\partial t^l)^{\alpha_l}$ at $\mathbf{t} = 0$. For $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}^n$, let $\partial_{\mathbf{y}}^{\mathbf{d}} := \partial^{|\mathbf{d}|} / (\partial y^1)^{d_1} \dots (\partial y^n)^{d_n}$, where $|\mathbf{d}| := d_1 + \dots + d_n$. Then, it follows from the chain rule and the Leibniz rule that

$$\partial_{\mathbf{t}, 0}^\alpha (f(h_T(\mathbf{x})))$$

is a summation over $\mathbb{Z}_{\geq 0}$ of terms of the following form

$$(\partial_{\mathbf{y}}^{\mathbf{d}} f)(h(\mathbf{x})) \cdot \partial_{\mathbf{t}, 0}^{\alpha_{i_1}} h_T^{i_1}(\mathbf{x}) \dots \partial_{\mathbf{t}, 0}^{\alpha_{i_I}} h_T^{i_I}(\mathbf{x})$$

with $|\mathbf{d}| \leq |\alpha|$, $\alpha_{i_1} + \dots + \alpha_{i_I} = \alpha$ and $1 \leq i_1 < \dots < i_I \leq n$, $1 \leq I \leq n$. In particular, it is a linear combination of such $\partial_{\mathbf{t}, 0}^{\alpha_{i_1}} h_T^{i_1}(\mathbf{x}) \dots \partial_{\mathbf{t}, 0}^{\alpha_{i_I}} h_T^{i_I}(\mathbf{x})$ with coefficients all depending universally on f and h alone (i.e. with coefficients not depending on how h is extended to h_T). This universal identity can be made precise as follows.

- Let $|\alpha| = k$. The fiberwise Taylor Theorem applied to f as a differentiable function on $(X \times Y)/X$ in a neighborhood of the locus $\{y^1 = h^1(\mathbf{x}), \dots, y^n = h^n(\mathbf{x})\} \subset X \times Y$ gives

$$\begin{aligned} f(\mathbf{y}) &= \sum_{d=0}^k \frac{1}{d!} \sum_{\mathbf{d}, |\mathbf{d}|=d} m_{\mathbf{d}} \cdot (\partial_{\mathbf{y}}^{\mathbf{d}} f)(h(\mathbf{x})) (\mathbf{y} - h(\mathbf{x}))^{\mathbf{d}} \\ &\quad + \frac{1}{(k+1)!} \sum_{\mathbf{d}, |\mathbf{d}|=k+1} m_{\mathbf{d}} \cdot Q_{\mathbf{d}}(h(\mathbf{x})) (\mathbf{y} - h(\mathbf{x}))^{\mathbf{d}} \end{aligned}$$

for some $Q_{\mathbf{d}} \in C^\infty(Y)$. Here, $m_{\mathbf{d}}$ is the multiplicity factor associated to \mathbf{d} and $(\mathbf{y} - h(\mathbf{x}))^{\mathbf{d}} := (y^1 - h^1(\mathbf{x}))^{d_1} \dots (y^n - h^n(\mathbf{x}))^{d_n}$.

• When $h = h_0$ is extended to h_T , then for $\mathbf{t} \in T$ close enough to 0,

$$\begin{aligned} f(h_{\mathbf{t}}(\mathbf{x})) &= f(\mathbf{y})|_{\mathbf{y} \rightsquigarrow h_{\mathbf{t}}(\mathbf{x})} \\ &= \sum_{d=0}^k \frac{1}{d!} \sum_{\mathbf{d}, |\mathbf{d}|=d} m_{\mathbf{d}} \cdot (\partial_{\mathbf{y}}^{\mathbf{d}} f)(h(\mathbf{x}))(h_{\mathbf{t}}(\mathbf{x}) - h(\mathbf{x}))^{\mathbf{d}} \\ &\quad + \frac{1}{(k+1)!} \sum_{\mathbf{d}, |\mathbf{d}|=k+1} m_{\mathbf{d}} \cdot Q_{\mathbf{d}}(h(\mathbf{x}))(h_{\mathbf{t}}(\mathbf{x}) - h(\mathbf{x}))^{\mathbf{d}}, \end{aligned}$$

where $(h_{\mathbf{t}}(\mathbf{x}) - h(\mathbf{x}))^{\mathbf{d}} := (h_{\mathbf{t}}^1(\mathbf{x}) - h^1(\mathbf{x}))^{d_1} \cdots (h_{\mathbf{t}}^n(\mathbf{x}) - h^n(\mathbf{x}))^{d_n}$, and, hence,

$$\partial_{\mathbf{t},0}^{\alpha}(f(h_T(\mathbf{x}))) = \sum_{d=0}^k \frac{1}{d!} \sum_{\mathbf{d}, |\mathbf{d}|=d} m_{\mathbf{d}} \cdot (\partial_{\mathbf{y}}^{\mathbf{d}} f)(h(\mathbf{x})) \cdot \partial_{\mathbf{t},0}^{\alpha} \left((h_{\mathbf{t}}(\mathbf{x}) - h(\mathbf{x}))^{\mathbf{d}} \right).$$

From this aspect, Proposition 4.2.3.1 is nothing but the equal of the above identity in our particular noncommutative situation, with the map $h : X \rightarrow Y$ replaced by the map $\varphi : (X^{Az}, E) \rightarrow Y$ and the extension h_T of h replaced by the extension of φ_T of φ .

However, caution that in the commutative situation, $\partial_{\mathbf{t},0}^{\alpha}(f(h_T(\mathbf{x})))$ involves only $\partial_{\mathbf{y}}^{\mathbf{d}} f$ along the graph of h up to (and including) order $|\alpha|$ (i.e. restriction of f to the $|\alpha|$ -th infinitesimal neighborhood of the graph of h) while in our noncommutative situation, $\partial_{\mathbf{t},0}^{\alpha}(\varphi_T^{\sharp}(f))$ may involve $\partial_{\mathbf{y}}^{\mathbf{d}} f$ along the support $Supp(\tilde{\mathcal{E}}_{\varphi}) \subset X \times Y$ of the graph $\tilde{\mathcal{E}}_{\varphi}$ of φ up to (and including) order $r|\alpha|$, where r is the rank of E as a complex vector bundle on X . The detail depends on the nilpotency of the structure sheaf $\mathcal{O}_{Supp(\tilde{\mathcal{E}}_{\varphi})}$ of $Supp(\tilde{\mathcal{E}}_{\varphi}) \subset X \times Y$.

Remark/Notation 4.2.3.5. [equivalent form: general order] (Cf. Remark/Notation 4.1.1.) Recall from the proof of Proposition 4.2.3.1 the expression

$$R^{f;q_j}[k] = \sum_{\mathbf{d}} c_{j,\mathbf{d}} \mathbf{y}^{\mathbf{d}} \in C^{\infty}(U'_f)[y^1, \dots, y^n].$$

As in Remark/Notation 4.1.1, define

$$R^f[k] = \sum_{\mathbf{d}} \left(\sum_{j=1}^s c_{j,\mathbf{d}} \cdot Id_{(E_T|_{U'_f}^{(j)})|_{U'_f}} \right) \cdot \mathbf{y}^{\mathbf{d}}.$$

Then Proposition 4.2.3.1, with Remark 4.2.3.3, can be stated equivalently as

$$\partial^{\alpha}(\varphi_T^{\sharp}(f)) = R^f[k]|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \partial^{\alpha}(\varphi_T^{\sharp}(\mathbf{y}^{\mathbf{d}}))} \quad \text{for } \alpha \text{ with } |\alpha| = k \in \mathbb{Z}_{\geq 0}.$$

This generalizes Remark/Notation 4.1.1.

Furthermore, for $k = 1$, recall Notation 4.2.1.7 and let ∂_t be any of $\partial_{t^1}, \dots, \partial_{t^l}$ and $\alpha \in \mathbb{Z}_{\geq 0}^l$ be associated to ∂_t . Then, one has the following expansion of $\partial_t(\varphi_T^{\sharp}(f))$, linearly in $(\partial_t \varphi_T^{\sharp}(y^1), \dots, \partial_t \varphi_T^{\sharp}(y^n))$:

$$\begin{aligned} \partial_t(\varphi_T^{\sharp}(f)) &= \sum_{i=1}^n \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} \sum_{\tilde{\pi} \in \tilde{P}tn(\alpha, d), i(\tilde{\pi}, \mathbf{d})=i} \\ &\quad ([\partial_{y^i}^{\tilde{\pi}}] R^f[1]_{(\mathbf{d})})^L (\varphi_T^{\sharp}(\mathbf{y})) \cdot \partial_t \varphi_T^{\sharp}(y^i) \cdot ([\partial_{y^i}^{\tilde{\pi}}] R^f[1]_{(\mathbf{d})})^R (\varphi_T^{\sharp}(\mathbf{y})), \end{aligned}$$

where $R^f[1]_{(\mathbf{d})}$ is the multi-degree- \mathbf{d} component of $R^f[1]$ as a polynomial in (y^1, \dots, y^n) and $(\dots)^{L,R}(\varphi_T^{\sharp}(\mathbf{y})) := (\dots)^{L,R}(\varphi_T^{\sharp}(y^1), \dots, \varphi_T^{\sharp}(y^n)) = (\dots)^{L,R}|_{\mathbf{y}^{\mathbf{d}'} \rightsquigarrow \varphi_T^{\sharp}(\mathbf{y}^{\mathbf{d}'})}$.

The following is an immediate consequence of Remark 4.2.3.2:

Corollary 4.2.3.6. [chain rule under trace] *Under the trace map $Tr : C^\infty(\text{End}_{\mathbb{C}}(E)) \rightarrow C^\infty(X)^{\mathbb{C}}$, the chain rule for a first-order derivation holds:*

$$Tr \left(\partial_{\bullet}(\varphi_T^{\#}(f)) \right) = Tr \left(\sum_{i=1}^n \frac{\partial f}{\partial y^i}(\varphi_T^{\#}(y^1), \dots, \varphi_T^{\#}(y^n)) \cdot \partial_{\bullet} \varphi_T^{\#}(y^i) \right),$$

where ∂_{\bullet} is any of $\partial/\partial t^1, \dots, \partial/\partial t^l$.

Proof. Let $\partial_{y_{\bullet}}$ be any of $\partial/\partial y^1, \dots, \partial/\partial y^n$. Then, with the notation in Remark/Notation 4.2.3.5, observe that, for $f \in C^\infty(Y)$,

$$\varphi_T^{\#}(\partial_{y_{\bullet}} f) = \varphi_T^{\#}(\partial_{y_{\bullet}}(R^f[1])) = (\partial_{y_{\bullet}}(R^f[1]))|_{\mathbf{y}^d \rightsquigarrow \varphi_T^{\#}(\mathbf{y}^d)} = R^f[1]|_{\mathbf{y}^d \rightsquigarrow \varphi_T^{\#}(\partial_{y_{\bullet}}(\mathbf{y}^d))}.$$

Since

$$Tr \left(\partial_{\bullet}(\varphi_T^{\#}(f)) \right) = Tr \left(\sum_{i=1}^n R^f[1]|_{\mathbf{y}^d \rightsquigarrow \varphi_T^{\#}(\partial_{y_i}(\mathbf{y}^d))} \partial_{\bullet} \varphi_T^{\#}(y^i) \right),$$

the corollary follows. □

5 The first variation of the Dirac-Born-Infeld action and the equations of motion for D-branes

We discuss in this section the first variation of the Dirac-Born-Infeld action (Sec. 5.2) and its consequence, the equations of motion of D-branes in our setting (Sec. 5.3). We begin with a few remarks on variations and infinitesimal deformations in C^∞ -algebraic geometry (Sec. 5.1).

5.1 Remark on deformation problems in C^∞ -algebraic geometry

From the viewpoint of C^∞ -algebraic geometry, it is very natural to address a deformation problem as an extension problem over a non-reduced C^∞ -scheme, as did in the setting Grothendieck's Modern Algebraic Geometry, e.g. [II], [Ser], [Schl]. On the other hand, for a variation problem in differential or symplectic geometry, it is customary to consider a 1- or 2-parameter family of objects in question and then take derivatives. The following very elementary example indicates that the former is more general than the latter:

Example 5.1.1. [infinitesimal extension vs. extension over $(-\delta, \delta) \subset \mathbb{R}^1$] Let

$$(p, \text{End}_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}, \mathbb{C}) \xrightarrow{\varphi} Y = \mathbb{R}^1$$

be a map from an Azumaya/matrix point of rank 1 (ie. a \mathbb{C} -point) to $Y = \mathbb{R}^1$, defined by a ring-homomorphism over $\mathbb{R} \subset \mathbb{C}$:

$$\begin{array}{ccc} \text{End}_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C} & \xleftarrow{\varphi^{\#}} & C^\infty(\mathbb{R}^1) \\ \lambda & \xleftarrow{\quad} & y \\ f(\lambda) & \xleftarrow{\quad} & f(y) \end{array}$$

for a $\lambda \in \mathbb{R} \subset \mathbb{C}$ fixed. Let $T_1 := \text{Spec}^{\mathbb{R}}(\mathbb{R}[t]/(t^2)) =: \text{Spec}^{\mathbb{R}}(\mathbb{R}[\epsilon])$, $\epsilon^2 = 0$, be a dual-point and $T_2 := (-\delta, \delta) \subset \mathbb{R}$ be a 1-manifold with parameter t , where $\delta > 0$ small. $T_1 \subset T_2$ as C^∞ -subscheme. Treat φ^\sharp as a ring-homomorphism over $\text{Spec}^{\mathbb{R}} \mathbb{R} \subset T_1$. Then, the following is an infinitesimal extension of φ^\sharp to a ring-homomorphism $\varphi_{T_1}^\sharp$ over the base T_1 :

$$\begin{array}{ccc} \text{End}_{\mathbb{C}[\epsilon]}(\mathbb{C}[\epsilon]) \simeq \mathbb{C}[\epsilon] & \xleftarrow{\varphi_{T_1}^\sharp} & C^\infty(\mathbb{R}^1) \\ \lambda + \sqrt{-1}\epsilon & \xleftarrow{\quad} & y \\ f(\lambda) + f'(\lambda)\sqrt{-1}\epsilon & \xleftarrow{\quad} & f(y) \end{array} .$$

On the other hand, let E_{T_2} be the trivialized complex line bundle over T_2 . Then, since T_2 is a manifold, any extension of $\varphi^\sharp =: \varphi_0^\sharp$ to a ring-homomorphism $\varphi_{T_2}^\sharp$ over the base T_2 must be of the following form

$$\begin{array}{ccc} C^\infty(\text{End}_{\mathbb{C}}(E_{T_2})) & \xleftarrow{\varphi_{T_2}^\sharp} & C^\infty(\mathbb{R}^1) \\ h(t) & \xleftarrow{\quad} & y \\ f(h(t)) & \xleftarrow{\quad} & f(y) \end{array} ,$$

for $t \in T_2$, where $h \in C^\infty(T_2)$ with $h(0) = \lambda$. Whose associated infinitesimal deformation of φ^\sharp is given by $\varphi_{T_2}^\sharp|_{T_1}$:

$$\begin{array}{ccc} \text{End}_{\mathbb{C}[\epsilon]}(\mathbb{C}[\epsilon]) \simeq \mathbb{C}[\epsilon] & \xleftarrow{\varphi_{T_2}^\sharp|_{T_1}} & C^\infty(\mathbb{R}^1) \\ \lambda + h'(0)\epsilon & \xleftarrow{\quad} & y \\ f(\lambda) + f'(\lambda)h'(0)\epsilon & \xleftarrow{\quad} & f(y) \end{array} .$$

Since $f'(\lambda)h'(0) \in \mathbb{R}$, this can never be the given $\varphi_{T_1}^\sharp$. In other words, $\varphi_{T_1}^\sharp$ cannot be extended further to a ring-homomorphism over $T_2 \supset T_1$.

The above example demonstrates the fact that there can be infinitesimal deformations in a moduli problem that do not arise from a smooth family. Such a phenomenon may be unfamiliar to differential geometers or string-theorists but is completely normal to algebraic geometers. It only means that the associated moduli stack is singular at the point representing that object in question and hence some infinitesimal deformations of that object can be obstructed from further extensions. From this point of view, our treatment of the variation problem below through a smooth family is not yet the most general one. But we will focus only on such unobstructed deformations for the current notes. The more general, possibly obstructed, deformations in our problem and their consequences should be understood better in the future.

5.2 The first variation of the Dirac-Born-Infeld action

Given an admissible Lorentzian map,

$$\varphi : (X^{\mathbb{A}^z}, E; \nabla) \longrightarrow (Y, g, B, \Phi),$$

let $T := (-\varepsilon, \varepsilon) \subset \mathbb{R}^1$ and $\varphi_t : (X^{\mathbb{A}^z}, E; \nabla^t) \rightarrow (Y, g, B, \Phi)$, $t \in T$, be a differentiable T -family of admissible Lorentzian maps that deforms $\varphi =: \varphi_0$. In this subsection we derive in steps the first variation

$$\left. \frac{d}{dt} \right|_{t=0} S_{DBI}^{(\Phi, g, B)}(\varphi_t, \nabla^t)$$

of the Dirac-Born-Infeld action. The derivation for the other two situations: (Y, g) Lorentzian and φ_t spacelike, and (Y, g) Riemannian and φ_t Riemannian, are completely the same.

As the major part of the discussion is local and around $0 \in T$, we will assume that ε is small enough and set the computation over a small enough coordinate chart $U \subset X$ (with coordinate functions $\mathbf{x} = (x^1, \dots, x^m)$) so that $E|_U$ is trivializable and trivialized, and $\varphi_t(U)$ is contained in a coordinate chart $V \subset Y$ (with coordinate functions $\mathbf{y} = (y^1, \dots, y^n)$). Recall from Sec. 3.2 that, over U ,

$$\begin{aligned} S_{DBI}^{(\Phi, g, B)}|_U(\varphi_t, \nabla^t) &= -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(e^{-\varphi_t^\diamond \Phi} \sqrt{-\operatorname{SymDet}_U(\varphi_t^\diamond(g+B) + 2\pi\alpha' F_{\nabla^t})} \right) \right) \\ &= -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(e^{-\varphi_t^\sharp(\Phi)} \sqrt{-\operatorname{SymDet} \left(\sum_{i,j} \varphi_t^\sharp(E_{ij}) D_\mu^t \varphi_t^\sharp(y^i) D_\nu^t \varphi_t^\sharp(y^j) + 2\pi\alpha' [\nabla_\mu^t, \nabla_\nu^t] \right)_{\mu\nu}} \right) \right) d^m \mathbf{x}. \end{aligned}$$

Here, we set the notation for the tensors and connections involved as follows:

- $g + B = \sum_{i,j} (g_{ij} + B_{ij}) dy^i \otimes dy^j =: \sum_{i,j} E_{ij} dy^i \otimes dy^j$, with $g_{ij} = g_{ji}$, $B_{ij} = -B_{ji}$,
- $\nabla^t = d + A^t = \sum_\mu (\partial_\mu + A_\mu^t) dx^\mu$ is the connection on $E|_U$,
- $D^t = d + [A^t, \cdot] = \sum_\mu (\partial_\mu + [A_\mu^t, \cdot]) dx^\mu$ is the ∇^t -induced connection on $\operatorname{End}_{\mathbb{C}}(E|_U)$,
- $d^m \mathbf{x} := dx^1 \wedge \dots \wedge dx^m$ is compatible with the orientation on U ;

and, for later use,

$$\varphi^\sharp(y^i) := \frac{d}{dt} \Big|_{t=0} \left(\varphi_T^\sharp(y^i) \right), \quad \cdot(\varphi^\sharp(\mathbf{y}^d)) := \frac{d}{dt} \Big|_{t=0} (\varphi_T^\sharp(\mathbf{y}^d)), \quad \dot{A}_\mu := \frac{d}{dt} \Big|_{t=0} A_\mu^T.$$

We assume further that the local chart U and $\varphi > 0$ are small enough so that the construction over $U_T := U \times (-\varepsilon, \varepsilon)$ in Sec. 4.1, with $p \in U \times \{0\} \subset U_T$, applies simultaneously to $e^{-\Phi}$ and E_{ij} , $i, j = 1, \dots, n$, to give the local expression of $\varphi_T^\sharp(\Phi)$ and $\varphi_T^\sharp(E_{ij})$, $i, j = 1 \dots, n$, in terms of elements in the polynomial ring over $C^\infty(U_T)$

$$\varphi_T^\sharp(\Phi), \varphi_T^\sharp(E_{ij}) \in \left(\oplus_{j=1}^s C^\infty(U_T) \cdot \operatorname{Id}_{E_T^{(j)}} \right) [\varphi_T^\sharp(y^1), \dots, \varphi_T^\sharp(y^n)]$$

of multi-degree $\leq (r-1, \dots, r-1)$. Associated to these settings and with the notation from Remark/Notataion 4.2.3.5, recall that

$$\begin{aligned} e^{-\varphi_T^\sharp(\Phi)} &= \varphi_T^\sharp(e^{-\Phi}) = R^{e^{-\Phi}}[0]_{\mathbf{y}^d \rightsquigarrow \varphi_T^\sharp(\mathbf{y}^d)}, \\ \frac{d}{dt} \Big|_{t=0} e^{-\varphi_T^\sharp(\Phi)} &= \frac{d}{dt} \Big|_{t=0} \varphi_T^\sharp(e^{-\Phi}) = R^{e^{-\Phi}}[1]_{\mathbf{y}^d \rightsquigarrow \cdot(\varphi_T^\sharp(\mathbf{y}^d))}; \end{aligned}$$

and

$$\begin{aligned} \varphi_T^\sharp(E_{ij}) &= R^{E_{ij}}[0]_{\mathbf{y}^d \rightsquigarrow \varphi_T^\sharp(\mathbf{y}^d)}, \\ \frac{d}{dt} \Big|_{t=0} \varphi_T^\sharp(E_{ij}) &= R^{E_{ij}}[1]_{\mathbf{y}^d \rightsquigarrow \cdot(\varphi_T^\sharp(\mathbf{y}^d))}, \end{aligned}$$

for $i, j = 1, \dots, n$. For simplicity of notation, it is understood that $R^{E_{ij}}[1]$ is evaluated at $t = 0$ in the expression $R^{E_{ij}}[1]_{\mathbf{y}^d \rightsquigarrow \cdot(\varphi_T^\sharp(\mathbf{y}^d))}$; and similarly for induced expressions that follow this.

Basic identities

Basic identities that will be used in the calculation are collected here for reference.

(a) *Differentiation of a square root* Let $M(t) \in C^\infty(\text{End}_{\mathbb{C}}(E))$, $t \in T := (-\varepsilon, \varepsilon) \subset \mathbb{R}$, be a T -family of invertible endomorphisms of E such that $\sqrt{M(t)}$ is well-defined, cf. Sec. 3.1.4. Denote $\frac{d}{dt}\big|_{t=0} M(t)$ by $\dot{M}(0)$. Then, $\sqrt{M(t)}$, $t \in (-\varepsilon, \varepsilon)$, is also invertible and

$$\sqrt{M(0)}^{-1} \left(\frac{d}{dt}\bigg|_{t=0} \sqrt{M(t)} \right) \sqrt{M(0)} + \left(\frac{d}{dt}\bigg|_{t=0} \sqrt{M(t)} \right) = \sqrt{M(0)}^{-1} \dot{M}(0).$$

It follows that

$$\text{Tr} \left(\frac{d}{dt}\bigg|_{t=0} \sqrt{M(t)} \right) = \frac{1}{2} \text{Tr} \left(\sqrt{M(0)}^{-1} \dot{M}(0) \right).$$

Slightly more generally, if $C \in C^\infty(\text{End}_{\mathbb{C}}(E))$ commutes with $\sqrt{M(0)}$, then

$$\sqrt{M(0)}^{-1} \left(C \cdot \frac{d}{dt}\bigg|_{t=0} \sqrt{M(t)} \right) \sqrt{M(0)} + \left(C \cdot \frac{d}{dt}\bigg|_{t=0} \sqrt{M(t)} \right) = C \cdot \sqrt{M(0)}^{-1} \dot{M}(0).$$

It follows that

$$\text{Tr} \left(C \cdot \frac{d}{dt}\bigg|_{t=0} \sqrt{M(t)} \right) = \frac{1}{2} \text{Tr} \left(C \cdot \sqrt{M(0)}^{-1} \dot{M}(0) \right).$$

(b) *Identities on symmetrized determinant and its differentiation* The Leibniz rule holds for a symmetric product:

$$\partial \cdot (a_1 \odot \cdots \odot a_m) = \sum_{\mu=1}^m a_1 \odot \cdots \odot a_{\mu-1} \odot (\partial \cdot a_\mu) \odot a_{\mu+1} \odot \cdots \odot a_m.$$

It follows that if let $M = [M^{(1)}, \dots, M^{(m)}]$ be the presentation of an $m \times m$ matrix M in terms of its column vectors, then

$$\partial \cdot \text{SymDet}(M) = \sum_{\nu=1}^m \text{SymDet}([M^{(1)}, \dots, M^{(\nu-1)}, \partial \cdot M^{(\nu)}, M^{(\nu+1)}, \dots, M^{(m)}]).$$

Similarly, for M presented in terms of its row vectors.

(c) *Trace and Lie bracket* For $r \times r$ matrices or matrix-valued functions A , B , and C ,

$$\text{Tr}(A[B, C]) = \text{Tr}([A, B]C).$$

(d) $\partial \cdot \text{Tr} = \text{Tr} D$. Recall the induced connection D on $\text{End}_{\mathbb{C}}(E)$ from ∇ on E .

• Let $s \in C^\infty(\text{End}_{\mathbb{C}}(E))$. Then $\partial \cdot \text{Tr}(s) = \text{Tr}(D \cdot s)$.

Proof. In any local presentation of E , let $\nabla = d + A$, where A is the $\text{End}_{\mathbb{C}}(E)$ -valued connection 1-form on X with respect to the local trivialization. Then $D = d + [A, \cdot]$ with respect to the induced local trivialization of $\text{End}_{\mathbb{C}}(E)$. It follows that

$$\text{Tr}(D \cdot (s)) = \text{Tr}(\partial \cdot s + [A, s]) = \text{Tr}(\partial \cdot s) = \partial \cdot \text{Tr}(s).$$

□

The first variation of each ingredient in the Dirac-Born-Infeld action

(a) *The first variation of $e^{-\varphi^\sharp(\Phi)}$ and $\varphi^\sharp(E_{ij})$* Recall Remark/Notation 4.2.3.5. Then, it follows from Proposition 4.2.3.1 that

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} \left(e^{-\varphi^\sharp(\Phi)} \right) &= R^{e^{-\Phi}}[1] \big|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \cdot (\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))} \\
&= \sum_{i'=1}^n \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} \sum_{\vec{\pi} \in P\vec{tn}(1,d), i_{(\vec{\pi}, \mathbf{d})}=i'} \\
&\quad ([\partial_{y^{i'}}^{\vec{\pi}}] R^{e^{-\Phi}}[1]_{(\mathbf{d})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^{i'}) \cdot ([\partial_{y^{i'}}^{\vec{\pi}}] R^{e^{-\Phi}}[1]_{(\mathbf{d})}^R(\varphi^\sharp(\mathbf{y}))) \\
&=: \sum_{i'=1}^n \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i_{(\vec{\pi}, \mathbf{d})}=i'} R^{e^{-\Phi}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^{i'}) \cdot R^{e^{-\Phi}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y}))
\end{aligned}$$

and

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} \left(\varphi_T^\sharp(E_{ij}) \right) &= R^{E_{ij}}[1] \big|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \cdot (\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))} \\
&= \sum_{i'=1}^n \sum_{d=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} \sum_{\vec{\pi} \in P\vec{tn}(1,d), i_{(\vec{\pi}, \mathbf{d})}=i'} \\
&\quad ([\partial_{y^{i'}}^{\vec{\pi}}] R^{E_{ij}}[1]_{(\mathbf{d})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^{i'}) \cdot ([\partial_{y^{i'}}^{\vec{\pi}}] R^{E_{ij}}[1]_{(\mathbf{d})}^R(\varphi^\sharp(\mathbf{y}))) \\
&=: \sum_{i'=1}^n \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i_{(\vec{\pi}, \mathbf{d})}=i'} R^{E_{ij}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^{i'}) \cdot R^{E_{ij}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})).
\end{aligned}$$

(b) *The first variation of $D_\mu \varphi^\sharp(y^i)$ and $F_{\mu\nu}$* By straightforward computation,

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} \left(D_\mu^T \varphi_T^\sharp(y^i) \right) &= \left. \frac{d}{dt} \right|_{t=0} \left(\partial_\mu \varphi_T^\sharp(y^i) + [A_\mu^T, \varphi_T^\sharp(y^i)] \right) \\
&= D_\mu \dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_\mu]
\end{aligned}$$

and

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} F_{\mu\nu}^T &= \left. \frac{d}{dt} \right|_{t=0} [\nabla_\mu^T, \nabla_\nu^T] \\
&= D_\mu \dot{A}_\nu - D_\nu \dot{A}_\mu.
\end{aligned}$$

The first variation of the Dirac-Born-Infeld action

With all the ingredients prepared, the computation of the first variation of $S_{DBI}(\varphi, \nabla)$ is now straightforward, though some of the expressions may look complicated due to noncommutativity. We proceed in five steps.

Step (1) : Input from all the pieces

Let

$$\mathbf{M}_{\mu\nu}(t) := \sum_{i,j} \varphi_t^\sharp(E_{ij}) D_\mu^t \varphi_t^\sharp(y^i) D_\nu^t \varphi_t^\sharp(y^j) + 2\pi\alpha' [\nabla_\mu^t, \nabla_\nu^t] \in C^\infty(\text{End}_{\mathbb{C}}(E|_U))$$

and $\mathbf{M}(t) := [\mathbf{M}_{\mu\nu}(t)]_{\mu\nu}$ the $m \times m$ matrix with (μ, ν) -entry $\mathbf{M}_{\mu\nu}(t)$. Then,

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} S_{DBI}(\varphi_t, \nabla^t) &= -T_{m-1} \frac{d}{dt} \Big|_{t=0} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(e^{-\varphi_t^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right) \right) d^m \mathbf{x} \\ &= -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \frac{d}{dt} \Big|_{t=0} \left(e^{-\varphi_t^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right) \right) d^m \mathbf{x}; \end{aligned}$$

$$\begin{aligned} \operatorname{Tr} \frac{d}{dt} \Big|_{t=0} \left(e^{-\varphi_t^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right) \\ = \operatorname{Tr} \left((R^{e^{-\Phi}}[1]_{t=0})|_{\mathbf{y}^I \rightsquigarrow \cdot_{(\varphi^\sharp(\mathbf{y}^I))}} \cdot \sqrt{-\operatorname{SymDet}(\mathbf{M}(0))} \right) + \operatorname{Tr} \left(e^{-\varphi^\sharp(\Phi)} \frac{d}{dt} \Big|_{t=0} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right). \end{aligned}$$

Since (φ, ∇) is admissible, $e^{-\varphi^\sharp(\Phi)}$ and $\sqrt{-\operatorname{SymDet}(\mathbf{M}(t))}$ commute. Thus,

$$\operatorname{Tr} \left(e^{-\varphi^\sharp(\Phi)} \frac{d}{dt} \Big|_{t=0} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right) = \frac{-1}{2} \operatorname{Tr} \left(e^{-\varphi^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(0))}^{-1} \cdot \frac{d}{dt} \Big|_{t=0} \operatorname{SymDet}(\mathbf{M}(t)) \right).$$

Denote by $[\cdot]^\top$ the transpose of the matrix $[\cdot]$ and let

$$\mathbf{M}(t) = \begin{bmatrix} \mathbf{M}_{(1)}(t) \\ \vdots \\ \mathbf{M}_{(m)}(t) \end{bmatrix} = \left[\mathbf{M}_{(1)}^\top, \dots, \mathbf{M}_{(m)}^\top \right]^\top$$

be the presentation of $\mathbf{M}(t)$ in terms of its row vectors and denote $\frac{d}{dt} \Big|_{t=0} \mathbf{M}_{(\mu)}(t)$ by $\dot{\mathbf{M}}_{(\mu)}(0)$, for $\mu = 1, \dots, m$. Then

$$\frac{d}{dt} \Big|_{t=0} \operatorname{SymDet}(\mathbf{M}(t)) = \sum_{\mu=1}^m \operatorname{SymDet}([\mathbf{M}_{(1)}(0)^\top, \dots, \mathbf{M}_{(\mu-1)}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\mu+1)}(0)^\top, \dots, \mathbf{M}_{(m)}(0)^\top]^\top).$$

Denote $\frac{d}{dt} \Big|_{t=0} \mathbf{M}_{\mu\nu}(t)$ by $\dot{\mathbf{M}}_{\mu\nu}(0)$, for $\mu, \nu = 1, \dots, m$. Then, the ν -th entry in $\dot{\mathbf{M}}_{(\mu)}(0)$ is given by

$$\begin{aligned} \dot{\mathbf{M}}_{\mu\nu}(0) &= \sum_{i,j} R^{E_{ij}}[1]|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \cdot_{(\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}} \cdot D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) \\ &\quad + \sum_{i,j} \varphi^\sharp(E_{ij}) \cdot (D_\mu \dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_\mu]) \cdot D_\nu \varphi^\sharp(y^j) \\ &\quad + \sum_{i,j} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \cdot (D_\nu \dot{\varphi}^\sharp(y^j) - [\varphi^\sharp(y^j), \dot{A}_\nu]) + 2\pi\alpha' (D_\mu \dot{A}_\nu - D_\nu \dot{A}_\mu). \end{aligned}$$

With

$$\mathbf{M}_{\mu\nu}(0) = \sum_{i,j} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) + 2\pi\alpha' [\nabla_\mu, \nabla_\nu],$$

one has altogether:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} S_{DBI}(\varphi_t, \nabla^t) &= -T_{m-1} \frac{d}{dt} \Big|_{t=0} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(e^{-\varphi_t^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right) \right) d^m \mathbf{x} \\ &= -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(R^{e^{-\Phi}}[1]|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \cdot_{(\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}} \cdot \sqrt{-\operatorname{SymDet}(\mathbf{M}(0))} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} e^{-\varphi^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(0))}^{-1} \right. \right. \\ &\quad \left. \left. \cdot \sum_{\mu=1}^m \operatorname{SymDet}([\mathbf{M}_{(1)}(0)^\top, \dots, \mathbf{M}_{(\mu-1)}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\mu+1)}(0)^\top, \dots, \mathbf{M}_{(m)}(0)^\top]^\top) \right) \right) d^m \mathbf{x} \\ &= -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \left((R^{e^{-\Phi}}[1])|_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \cdot_{(\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}} \cdot \sqrt{-\operatorname{SymDet}(\mathbf{M}(0))} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} e^{-\varphi^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(0))}^{-1} \right. \right. \\ &\quad \left. \left. \cdot \sum_{\mu=1}^m \sum_{\sigma \in \operatorname{Sym}_m} (-1)^\sigma \mathbf{M}_{1\sigma(1)}(0) \odot \dots \odot \mathbf{M}_{(\mu-1)\sigma(\mu-1)}(0) \right. \right. \\ &\quad \left. \left. \odot \dot{\mathbf{M}}_{\mu\sigma(\mu)}(0) \odot \mathbf{M}_{(\mu+1)\sigma(\mu+1)}(0) \odot \dots \odot \mathbf{M}_{m\sigma(m)}(0) \right) \right) d^m \mathbf{x}. \end{aligned}$$

Step (2) : Arrangement to boundary terms and the linear functional $\delta S_{DBI}(\varphi, \nabla)/\delta(\varphi, \nabla)$ on $(\dot{\varphi}^\sharp(y^1), \dots, \dot{\varphi}^\sharp(y^n); \dot{A}_1, \dots, \dot{A}_m)$

Summands from the first cluster

$$R^{e^{-\Phi}}[1]|_{\mathbf{y}^{\mathbf{d} \rightsquigarrow \cdot (\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}} \cdot \sqrt{-\text{SymDet}(\mathbf{M}(0))}$$

contain only $\dot{\varphi}^\sharp(y^i)$, $i = 1, \dots, n$, from $(R^{e^{-\Phi}}[1])|_{\mathbf{y}^{\mathbf{d} \rightsquigarrow \cdot (\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}}$. Hence, it contributes solely to the linear functional $\delta S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)/\delta(\varphi, \nabla)$ on $(\dot{\varphi}^\sharp(y^1), \dots, \dot{\varphi}^\sharp(y^n); \dot{A}_1, \dots, \dot{A}_m)$ and, hence, to the equations of motion for (φ, ∇) .

On the other hand, summands from the expansion of the second cluster

$$\begin{aligned} & -\frac{1}{2} e^{-\varphi^\sharp(\Phi)} \sqrt{-\text{SymDet}(\mathbf{M}(0))}^{-1} \\ & \cdot \sum_{\mu=1}^m \sum_{\sigma \in \text{Sym}_m} (-1)^\sigma \mathbf{M}_{1\sigma(1)}(0) \odot \dots \odot \mathbf{M}_{(\mu-1)\sigma(\mu-1)}(0) \\ & \odot \dot{\mathbf{M}}_{\mu\sigma(\mu)}(0) \odot \mathbf{M}_{(\mu+1)\sigma(\mu+1)}(0) \odot \dots \odot \mathbf{M}_{m\sigma(m)}(0) \end{aligned}$$

are of two types:

- One contains a factor in the list $\dot{\varphi}^\sharp(y^i)$, $i = 1, \dots, n$, \dot{A}_μ , $\mu = 1, \dots, m$ from some $\dot{\mathbf{M}}_{\mu'\nu'}(0)$, $\mu', \nu' = 1, \dots, m$. They contribute to the linear functional $\delta S_{DBI}(\varphi, \nabla)/\delta(\varphi, \nabla)$ on $(\dot{\varphi}^\sharp(y^1), \dots, \dot{\varphi}^\sharp(y^n); \dot{A}_1, \dots, \dot{A}_m)$ and, hence, to the equations of motion for (φ, ∇) .
- The other contains a factor in the list $D_\mu \dot{\varphi}(y^i)$, $i = 1, \dots, n$, $\mu = 1, \dots, m$, $D_\mu \dot{A}_\nu$, $\mu, \nu = 1, \dots, m$, from some $\dot{\mathbf{M}}_{\mu'\nu'}(0)$, $\mu', \nu' = 1, \dots, m$. After integration by parts, each contributes a boundary term in an integral $\int_{\partial U}(\dots)$ and a term in the linear functional $\delta S_{DBI}(\varphi, \nabla)/\delta(\varphi, \nabla)$ on $(\dot{\varphi}^\sharp(y^1), \dots, \dot{\varphi}^\sharp(y^n); \dot{A}_1, \dots, \dot{A}_m)$. The latter contributes then to the equations of motion for (φ, ∇) .

We now proceed to study their details.

Step (3) : Details for the first cluster

For the first cluster,

$$\begin{aligned} & \text{Tr} \left(R^{e^{-\Phi}}[1]|_{\mathbf{y}^{\mathbf{d} \rightsquigarrow \cdot (\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}} \cdot \sqrt{-\text{SymDet}(\mathbf{M}(0))} \right) \\ & = \text{Tr} \left(\left(\sum_{i'=1}^n \sum_{\mathbf{d}, \mathbf{d}, \bar{\pi}; |\mathbf{d}|=d, i_{(\bar{\pi}, \mathbf{d})}=i'} R^{e^{-\Phi}}[1]_{(\mathbf{d}, \bar{\pi})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^{i'}) \cdot R^{e^{-\Phi}}[1]_{(\mathbf{d}, \bar{\pi})}^R(\varphi^\sharp(\mathbf{y})) \right) \cdot \sqrt{-\text{SymDet}(\mathbf{M}(0))} \right) \\ & = \text{Tr} \left(\sum_{i'=1}^n \left(\sum_{\mathbf{d}, \mathbf{d}, \bar{\pi}; |\mathbf{d}|=d, i_{(\bar{\pi}, \mathbf{d})}=i'} R^{e^{-\Phi}}[1]_{(\mathbf{d}, \bar{\pi})}^R(\varphi^\sharp(\mathbf{y})) \cdot \sqrt{-\text{SymDet}(\mathbf{M}(0))} \cdot R^{e^{-\Phi}}[1]_{(\mathbf{d}, \bar{\pi})}^L(\varphi^\sharp(\mathbf{y})) \right) \cdot \dot{\varphi}^\sharp(y^{i'}) \right) \\ & =: \text{Tr} \left(\sum_{i'=1}^n \mathcal{N}_{i'}^{1, (\Phi, g, B)}(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^{i'}) \right). \end{aligned}$$

Step (4) : Details for the second cluster

For the second cluster, recall Lemma 3.1.3.8. Then

$$\begin{aligned} & \text{SymDet}([\mathbf{M}_{(1)}(0)^\top, \dots, \mathbf{M}_{(\mu-1)}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\mu+1)}(0)^\top, \dots, \mathbf{M}_{(m)}(0)^\top]^\top) = \\ & \frac{1}{m!} \sum_{\mu'=1}^m \sum_{\substack{\sigma \in \text{Sym}_m \\ \sigma(\mu') = \mu}} (-1)^\sigma \text{Det}([\mathbf{M}_{(\sigma(1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(\mu'-1))}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\sigma(\mu'+1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(m))}(0)^\top]^\top). \end{aligned}$$

Thus, denoting the factor $-\frac{1}{2} e^{-\varphi^\sharp(\Phi)} \sqrt{-\text{SymDet}(\mathbf{M}(0))}^{-1}$ by $F_2(\varphi, \nabla; \Phi, g, B)$,

$$\begin{aligned}
& \text{Tr} \left(-\frac{1}{2} e^{-\varphi^\sharp(\Phi)} \sqrt{-\text{SymDet}(\mathbf{M}(0))}^{-1} \right. \\
& \quad \cdot \sum_{\mu=1}^m \text{SymDet}([\mathbf{M}_{(1)}(0)^\top, \dots, \mathbf{M}_{(\mu-1)}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\mu+1)}(0)^\top, \dots, \mathbf{M}_{(m)}(0)^\top]^\top) \Big) \\
&= \text{Tr} \left(F_2(\varphi, \nabla; \Phi, g, B) \cdot \sum_{\mu=1}^m \text{SymDet}([\mathbf{M}_{(1)}(0)^\top, \dots, \mathbf{M}_{(\mu-1)}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\mu+1)}(0)^\top, \dots, \mathbf{M}_{(m)}(0)^\top]^\top) \right) \\
&= \text{Tr} \left(\frac{1}{m!} F_2(\varphi, \nabla; \Phi, g, B) \cdot \sum_{\mu=1}^m \sum_{\mu'=1}^m \sum_{\substack{\sigma \in \text{Sym}_m \\ \sigma(\mu') = \mu}} (-1)^\sigma \text{Det}([\mathbf{M}_{(\sigma(1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(\mu'-1))}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\sigma(\mu'+1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(m))}(0)^\top]^\top) \right) \\
&= \text{Tr} \left(\frac{1}{m!} \sum_{\mu=1}^m \sum_{\mu'=1}^m \sum_{\substack{\sigma \in \text{Sym}_m \\ \sigma(\mu') = \mu}} (-1)^\sigma \right. \\
& \quad \cdot \text{Det}([F_2(\varphi, \nabla; \Phi, g, B) \mathbf{M}_{(\sigma(1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(\mu'-1))}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top, \mathbf{M}_{(\sigma(\mu'+1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(m))}(0)^\top]^\top) \Big) \\
&= \text{Tr} \left(\frac{1}{m!} \sum_{\mu=1}^m \sum_{\mu'=1}^m \sum_{\substack{\sigma \in \text{Sym}_m \\ \sigma(\mu') = \mu}} (-1)^\sigma (-1)^{\mu'(m-\mu')} \right. \\
& \quad \cdot \text{Det}([\mathbf{M}_{(\sigma(\mu'+1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(m))}(0)^\top, F_2(\varphi, \nabla; \Phi, g, B) \mathbf{M}_{(\sigma(1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(\mu'-1))}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top]^\top) \Big) \\
& \quad \text{(by the invariance of trace under cyclic permutations)}.
\end{aligned}$$

Note that $\dot{\mathbf{M}}_{\mu\nu}(0)$, $\mu, \nu = 1, \dots, m$, now appear uniformly as the last factor in the summands from the expansion of $\text{Det}([\dots]^\top)$ above. Let $\text{Minor}(\varphi, \nabla; \Phi, g, B | \mu', \sigma)_{\mu\nu}$ be the (m, ν) -minor of $[\mathbf{M}_{(\sigma(\mu'+1))}(0), \dots, \mathbf{M}_{(\sigma(m))}(0), F_2(\varphi, \nabla; \Phi, g, B) \mathbf{M}_{(\sigma(1))}(0), \dots, \mathbf{M}_{(\sigma(\mu'-1))}(0), \dot{\mathbf{M}}_{(\mu)}(0)]^\top$. Then:

$$\begin{aligned}
&= \text{Tr} \left(\frac{1}{m!} \sum_{\mu=1}^m \sum_{\mu'=1}^m \sum_{\substack{\sigma \in \text{Sym}_m \\ \sigma(\mu') = \mu}} (-1)^\sigma (-1)^{\mu'(m-\mu')} \cdot \sum_{\nu=1}^m (-1)^{m+\nu} \text{Minor}(\varphi, \nabla; \Phi, g, B | \mu', \sigma)_{\mu\nu} \dot{\mathbf{M}}_{\mu\nu}(0) \right) \\
&= \text{Tr} \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \dot{\mathbf{M}}_{\mu\nu}(0) \right), \\
& \quad \text{where } \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \\
& \quad := \frac{1}{m!} \sum_{\mu'=1}^m \sum_{\substack{\sigma \in \text{Sym}_m \\ \sigma(\mu') = \mu}} (-1)^\sigma (-1)^{\mu'(m-\mu')+m+\nu} \text{Minor}(\varphi, \nabla; \Phi, g, B | \mu', \sigma)_{\mu\nu}, \\
&= \text{Tr} \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \right. \\
& \quad \cdot \left(\sum_{i,j} R^{E_{ij}}[1] |_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow \cdot_{(\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))}} \cdot D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) \right. \\
& \quad + \sum_{i,j} \varphi^\sharp(E_{ij}) \cdot (D_\mu \varphi^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_\mu]) \cdot D_\nu \varphi^\sharp(y^j) \\
& \quad + \sum_{i,j} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \cdot (D_\nu \varphi^\sharp(y^j) - [\varphi^\sharp(y^j), \dot{A}_\nu]) + 2\pi\alpha' (D_\mu \dot{A}_\nu - D_\nu \dot{A}_\mu) \Big) \Big) \\
&= \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} \quad \text{(defined in Step (4.1) – Step (4.4) below)}.
\end{aligned}$$

Let us now study each of the four subclusters of the second cluster separately.

Step (4.1) : The subcluster (I)

$$\begin{aligned}
(I) &:= \text{Tr} \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \cdot \sum_{i,j} R^{E_{ij}}[1]_{\mathbf{y}\mathbf{d} \rightsquigarrow (\varphi^\sharp(\mathbf{y}\mathbf{d}))} \cdot D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) \right) \\
&= \text{Tr} \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \right. \\
&\quad \cdot \sum_{i,j} \left(\sum_{i'=1}^n \sum_{d, \mathbf{d}, \bar{\pi}; |\mathbf{d}|=d, i_{(\bar{\pi}, \mathbf{d})}=i'} R^{E_{ij}}[1]_{(\mathbf{d}, \bar{\pi})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^{i'}) \cdot R^{E_{ij}}[1]_{(\mathbf{d}, \bar{\pi})}^R(\varphi^\sharp(\mathbf{y})) \right) \cdot D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) \Big) \\
&= \text{Tr} \left(\sum_{i'=1}^n \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \sum_{i,j} \sum_{d, \mathbf{d}, \bar{\pi}; |\mathbf{d}|=d, i_{(\bar{\pi}, \mathbf{d})}=i'} R^{E_{ij}}[1]_{(\mathbf{d}, \bar{\pi})}^R(\varphi^\sharp(\mathbf{y})) \cdot D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) \right. \right. \\
&\quad \left. \left. \cdot \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \cdot R^{E_{ij}}[1]_{(\mathbf{d}, \bar{\pi})}^L(\varphi^\sharp(\mathbf{y})) \right) \cdot \dot{\varphi}^\sharp(y^{i'}) \right) \\
&=: \text{Tr} \left(\sum_{i'=1}^n \mathcal{NL}_{i'}^{2.I, (\Phi, g, B)}(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^{i'}) \right).
\end{aligned}$$

Step (4.2) : The subcluster (II)

This subcluster contributes also to boundary terms.

$$\begin{aligned}
(II) &:= \text{Tr} \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \right. \\
&\quad \cdot \left(\sum_{i,j} \varphi^\sharp(E_{ij}) \cdot D_\mu \dot{\varphi}^\sharp(y^i) \cdot D_\nu \varphi^\sharp(y^j) + \sum_{i,j} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \cdot D_\nu \dot{\varphi}^\sharp(y^j) \right) \Big) \\
&= \text{Tr} \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \sum_{i,j} \left(D_\mu \varphi^\sharp(y^j) \cdot \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ji}) \right. \right. \\
&\quad \left. \left. + \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \right) \cdot D_\nu \dot{\varphi}^\sharp(y^j) \right) \\
&= \text{Tr} \left(\sum_{\nu=1}^m \sum_{\mu=1}^m \sum_{i,j} D_\nu \left[\left(D_\mu \varphi^\sharp(y^i) \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ji}) \right. \right. \right. \\
&\quad \left. \left. + \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \right) \dot{\varphi}^\sharp(y^j) \right] \Big) \\
&\quad - \text{Tr} \left(\sum_{j=1}^n \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \sum_{i=1}^n D_\nu \left[D_\mu \varphi^\sharp(y^i) \cdot \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ji}) \right. \right. \right. \\
&\quad \left. \left. + \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \right] \right) \cdot \dot{\varphi}^\sharp(y^j) \Big) \\
&= \sum_{\nu=1}^m \partial_\nu \text{Tr} \left(\sum_{\mu=1}^m \sum_{i,j} \left[\left(D_\mu \varphi^\sharp(y^i) \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ji}) \right. \right. \right. \\
&\quad \left. \left. + \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \right) \dot{\varphi}^\sharp(y^j) \right] \Big) \\
&\quad - \text{Tr} \left(\sum_{j=1}^n \left(\sum_{\mu=1}^m \sum_{\nu=1}^m \sum_{i=1}^n D_\nu \left[D_\mu \varphi^\sharp(y^i) \cdot \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ji}) \right. \right. \right. \\
&\quad \left. \left. + \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \right] \right) \cdot \dot{\varphi}^\sharp(y^j) \Big) \\
&=: \sum_{\nu=1}^m (-1)^{\nu-1} \partial_\nu (B\Gamma_\nu^{2.II, (\varphi, \nabla; \Phi, g, B)}(\dot{\varphi}^\sharp(\mathbf{y}))) + \text{Tr} \left(\sum_{j=1}^n \mathcal{NL}_j^{2.II, (\Phi, g, B)}(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^j) \right).
\end{aligned}$$

Step (4.3) : The subcluster (III)

$$\begin{aligned}
\text{(III)} &:= \text{Tr} \left(- \sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \right. \\
&\quad \cdot \left(\sum_{i,j} \varphi^\sharp(E_{ij}) \cdot [\varphi^\sharp(y^i), \dot{A}_\mu] \cdot D_\nu \varphi^\sharp(y^j) + \sum_{i,j} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \cdot [\varphi^\sharp(y^j), \dot{A}_\nu] \right) \Big) \\
&= \text{Tr} \left(\sum_{\nu=1}^m \sum_{\mu=1}^m \sum_{i,j} \left(\text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \varphi^\sharp(y^j) \right. \right. \\
&\quad - \varphi^\sharp(y^j) \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \\
&\quad - D_\mu \varphi^\sharp(y^j) \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ij}) \varphi^\sharp(y^i) \\
&\quad \left. \left. + \varphi^\sharp(y^i) D_\mu \varphi^\sharp(y^j) \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ij}) \right) \cdot \dot{A}_\nu \right) \\
&=: \text{Tr} \left(\sum_{\nu=1}^m \mathbf{NL}_\nu^{2,III, (\Phi, g, B)}(\varphi, \nabla) \cdot \dot{A}_\nu \right).
\end{aligned}$$

Step (4.4) : The subcluster (IV)

This subcluster contributes also to boundary terms.

$$\begin{aligned}
\text{(IV)} &:= \text{Tr} \left(2\pi\alpha' \sum_{\mu=1}^m \sum_{\nu=1}^m \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \cdot (D_\mu \dot{A}_\nu - D_\nu \dot{A}_\mu) \right) \\
&= \text{Tr} \left(2\pi\alpha' \sum_{\mu=1}^m \sum_{\nu=1}^m \left(\text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} - \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \right) \cdot D_\mu \dot{A}_\nu \right) \\
&= \text{Tr} \left(2\pi\alpha' \sum_{\mu=1}^m \sum_{\nu=1}^m D_\mu \left[\left(\text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} - \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \right) \cdot \dot{A}_\nu \right] \right) \\
&\quad - \text{Tr} \left(2\pi\alpha' \sum_{\nu=1}^m \left(\sum_{\mu=1}^m D_\mu \left[\text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} - \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \right] \right) \cdot \dot{A}_\nu \right) \\
&= \sum_{\mu=1}^m \partial_\mu \text{Tr} \left(2\pi\alpha' \sum_{\nu=1}^m \left(\text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} - \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \right) \cdot \dot{A}_\nu \right) \\
&\quad - \text{Tr} \left(2\pi\alpha' \sum_{\nu=1}^m \left(\sum_{\mu=1}^m D_\mu \left[\text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\mu\nu} - \text{ComboMinor}(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \right] \right) \cdot \dot{A}_\nu \right) \\
&=: \sum_{\mu=1}^m (-1)^{\mu-1} \partial_\mu (B \Gamma_\mu^{2,IV, (\varphi, \nabla; \Phi, g, B)}(\dot{A})) + \text{Tr} \left(\sum_{\nu=1}^m \mathbf{NL}_\nu^{2,IV, (\Phi, g, B)}(\varphi, \nabla) \cdot \dot{A}_\nu \right).
\end{aligned}$$

Step (5) : The final formula

In summary, with the notation introduced for the various nonlinear first-order and second-order differential expressions on (φ, ∇) that depend on (Φ, g, B) and appear in the calculation (subject to a relabelling of the dummy i' index), one has

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} S_{DBI}(\varphi_t, \nabla^t) &= -T_{m-1} \left. \frac{d}{dt} \right|_{t=0} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(e^{-\varphi_t^\sharp(\Phi)} \sqrt{-\operatorname{SymDet}(\mathbf{M}(t))} \right) \right) d^m \mathbf{x} \\
&= -T_{m-1} \int_U \operatorname{Re} \left(\sum_{\mu=1}^m (-1)^{\mu-1} \partial_\mu \left(B\Gamma_\mu^{2.II, (\varphi, \nabla; \Phi, g, B)}(\varphi^\sharp(\mathbf{y})) + B\Gamma_\mu^{2.IV, (\varphi, \nabla; \Phi, g, B)}(\dot{\mathbf{A}}) \right) \right) d^m \mathbf{x} \\
&\quad -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(\sum_{j=1}^n (\mathbf{N}\mathcal{L}_j^{1, (\Phi, g, B)}(\varphi, \nabla) + \mathbf{N}\mathcal{L}_j^{2.I, (\Phi, g, B)}(\varphi, \nabla) + \mathbf{N}\mathcal{L}_j^{2.II, (\Phi, g, B)}(\varphi, \nabla)) \cdot \varphi^\sharp(y^j) \right. \right. \\
&\quad \left. \left. + \sum_{\nu=1}^m (\mathbf{N}\mathcal{L}_\nu^{2.III, (\Phi, g, B)}(\varphi, \nabla) + \mathbf{N}\mathcal{L}_\nu^{2.IV, (\Phi, g, B)}(\varphi, \nabla)) \cdot \dot{A}_\nu \right) \right) d^m \mathbf{x} \\
&=: -T_{m-1} \int_{\partial U} \operatorname{Re} (B\Gamma^{(\varphi, \nabla; \Phi, g, B)}(\varphi^\sharp(\mathbf{y}), \dot{\mathbf{A}})) \\
&\quad -T_{m-1} \int_U \operatorname{Re} \left(\operatorname{Tr} \left(\sum_{j=1}^n \mathbf{N}\mathcal{L}_j^{(\Phi, g, B); \delta\varphi}(\varphi, \nabla) \cdot \varphi^\sharp(y^j) + \sum_{\nu=1}^m \mathbf{N}\mathcal{L}_\nu^{(\Phi, g, B); \delta\nabla}(\varphi, \nabla) \cdot \dot{A}_\nu \right) \right) d^m \mathbf{x}.
\end{aligned}$$

Here,

$$\begin{aligned}
&B\Gamma^{(\varphi, \nabla; \Phi, g, B)}(\varphi^\sharp(\mathbf{y}), \dot{\mathbf{A}}) \\
&:= \sum_{\mu=1}^m \left(B\Gamma_\mu^{2.II, (\varphi, \nabla; \Phi, g, B)}(\varphi^\sharp(\mathbf{y})) + B\Gamma_\mu^{2.IV, (\varphi, \nabla; \Phi, g, B)}(\dot{\mathbf{A}}) \right) dx^1 \wedge \cdots \wedge dx^{\mu-1} \wedge \widehat{dx^\mu} \wedge dx^{\mu+1} \cdots \wedge dx^m,
\end{aligned}$$

with the $\widehat{dx^\mu}$ meaning the removal of dx^μ , is a complex-valued $(m-1)$ -form on U that depends linearly on $(\mathbf{y}, \dot{\mathbf{A}})$ and whose real part gives the total boundary term (up to the factor $-T_{m-1}$) of the first variation of $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$ with respect to (φ, ∇) .

5.3 The equations of motion for D-branes

Remark 5.3.1. [effect of $\operatorname{Re}(\cdot)$ in action to equations of motion] Due to the operation ‘Taking the real part of’ $\operatorname{Re}(\cdot)$, to go from the the first variation formula to the expression for the equations of motion there is a detail that depends on how the space of pairs (φ, ∇) and its tangents $(\delta\varphi, \delta\nabla)$ are parameterized; (cf. $\operatorname{Re}(e^{\sqrt{-1}\theta} z) = \cos \theta \cdot \operatorname{Re}(z) - \sin \theta \cdot \operatorname{Im}(z)$).

- (1) For the φ -part, first, caution that it is *not* that just because $\varphi^\sharp(y^i)$, $i = 1, \dots, n$, take values in a ring over \mathbb{C} (i.e. $C^\infty(\operatorname{End}_{\mathbb{C}}(E))$) that the space $\operatorname{Map}((X^{A_z}, E), Y)$ of all such φ ’s becomes a complex space. Indeed, due to the fact that all the eigenvalues of $\varphi^\sharp(f)$, $f \in C^\infty(Y)$ are real (cf. [L-Y4: Sec. 3], D(11.1)), $\operatorname{Map}((X^{A_z}, E), Y)$ is intrinsically a real space and there is no natural complex-space structure on it (even if exists) that can be made compatible with the underlying moduli problem since if $\delta\varphi$ is an unobstructed tangent to $\operatorname{Map}((X^{A_z}, E), Y)$, then $\sqrt{-1}\delta\varphi$ can never be an unobstructed tangent to $\operatorname{Map}((X^{A_z}, E), Y)$. So this part is good in the sense that if we fix a real presentation for φ ’s in the study, then $\operatorname{Re}(\delta S_{DBI}/\delta\varphi)$ gives the system of equations of motion for φ .
- (2) For the ∇ -part, if alone, the parameter space is complex in nature in our most general setting. When E is Hermitian and ∇ is required to be compatible with the Hermitian structure, the resulting parameter space becomes intrinsically real. In the latter case, depending on the convention in presenting a unitary gauge theory (mathematicians vs. physicists), one may take either $\operatorname{Re}(\delta S_{DBI}^{(\Phi, g, B)}/\delta\nabla)$ or $\operatorname{Im}(\delta S_{DBI}^{(\Phi, g, B)}/\delta\nabla)$ as the system of equations for ∇ . *However*, this is not the full story as we imposed the admissible condition $\nabla_\bullet \mathcal{A}_\varphi \subset \mathcal{A}_\varphi$ on ∇ . Details on writing the equations of motion will have to depend on how we present this condition.

Not to let this additional detail to distract us in this first work in the D(13) subseries, we present for the current notes the system of equations of motion that remove the effect of $Re(\cdot)$ in $S_{DBI}^{(\Phi,g,B)}$. In other words, a true system of equations of motion will involve only a combination of what are given below.

It follows from the study in Sec. 5.2 that the equations of motion for D-branes from the Dirac-Born-Infeld action, with the D-brane world-volume modelled in the current context as an admissible map

$$\varphi : (X^{Az}, E; \nabla) \longrightarrow (Y, \Phi, g, B)$$

from an Azumaya/matrix manifold with a fundamental module with a connection $(X^{Az}, E; \nabla)$ to a space-time Y with massless background fields (Φ, g, B) from closed string excitations, are given by the following system of second-order nonlinear partial differential equations on (φ, ∇) :

$$\begin{cases} \mathcal{NL}_j^{(\Phi,g,B);\delta\varphi}(\varphi, \nabla) = 0, & \text{for } j = 1, \dots, n; \\ \mathcal{NL}_\nu^{(\Phi,g,B);\delta\nabla}(\varphi, \nabla) = 0, & \text{for } \nu = 1, \dots, m. \end{cases}$$

Here, for the first subsystem,

$$\mathcal{NL}_j^{(\Phi,g,B);\delta\varphi}(\varphi, \nabla) = \mathcal{NL}_j^{1,(\Phi,g,B)}(\varphi, \nabla) + \mathcal{NL}_j^{2.I,(\Phi,g,B)}(\varphi, \nabla) + \mathcal{NL}_j^{2.II,(\Phi,g,B)}(\varphi, \nabla)$$

with

$$\begin{aligned} \mathcal{NL}_j^{1,(\Phi,g,B)}(\varphi, \nabla) &= \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i_{(\vec{\pi}, \mathbf{d})}=j} R^{e^{-\Phi}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})) \cdot \sqrt{-SymDet(\mathbf{M}(0))} \cdot R^{e^{-\Phi}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})), \\ \mathcal{NL}_j^{2.I,(\Phi,g,B)}(\varphi, \nabla) &= \sum_{\mu=1}^m \sum_{\nu=1}^m \sum_{i', j'} \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i_{(\vec{\pi}, \mathbf{d})}=j} R^{E_{i'j'}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})) \cdot D_\mu \varphi^\sharp(y^i) D_\nu \varphi^\sharp(y^j) \\ &\quad \cdot ComboMinor(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \cdot R^{E_{i'j'}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})), \\ \mathcal{NL}_j^{2.II,(\Phi,g,B)}(\varphi, \nabla) &= - \sum_{\mu=1}^m \sum_{\nu=1}^m \sum_{i=1}^n D_\nu \left(D_\mu \varphi^\sharp(y^i) \cdot ComboMinor(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ji}) \right. \\ &\quad \left. + ComboMinor(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \right); \end{aligned}$$

and, for the second subsystem,

$$\mathcal{NL}_\nu^{(\Phi,g,B);\delta\nabla}(\varphi, \nabla) = \mathcal{NL}_\nu^{2.III,(\Phi,g,B)}(\varphi, \nabla) + \mathcal{NL}_\nu^{2.IV,(\Phi,g,B)}(\varphi, \nabla)$$

with

$$\begin{aligned} \mathcal{NL}_\nu^{2.III,(\Phi,g,B)}(\varphi, \nabla) &= \sum_{\mu=1}^m \sum_{i,j} \left(ComboMinor(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \varphi^\sharp(y^j) \right. \\ &\quad - \varphi^\sharp(y^j) ComboMinor(\varphi, \nabla; \Phi, g, B)_{\mu\nu} \varphi^\sharp(E_{ij}) D_\mu \varphi^\sharp(y^i) \\ &\quad - D_\mu \varphi^\sharp(y^j) ComboMinor(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ij}) \varphi^\sharp(y^i) \\ &\quad \left. + \varphi^\sharp(y^i) D_\mu \varphi^\sharp(y^j) ComboMinor(\varphi, \nabla; \Phi, g, B)_{\nu\mu} \varphi^\sharp(E_{ij}) \right), \\ \mathcal{NL}_\nu^{2.IV,(\Phi,g,B)}(\varphi, \nabla) &= 2\pi\alpha' \sum_{\mu=1}^m D_\mu (ComboMinor(\varphi, \nabla; \Phi, g, B)_{\nu\mu} - ComboMinor(\varphi, \nabla; \Phi, g, B)_{\mu\nu}). \end{aligned}$$

In both subsystems,

$$ComboMinor(\varphi, \nabla; \Phi, g, B)_{\mu\nu} = \frac{1}{m!} \sum_{\mu'=1}^m \sum_{\substack{\sigma \in Sym_m \\ \sigma(\mu') = \mu}} (-1)^\sigma (-1)^{\mu'(m-\mu')+m+\nu} Minor(\varphi, \nabla; \Phi, g, B | \mu', \sigma)_{\mu\nu},$$

where

$$Minor(\varphi, \nabla; \Phi, g, B | \mu', \sigma)_{\mu\nu} = \text{the } (m, \nu)\text{-minor of } [\mathbf{M}_{(\sigma(\mu'+1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(m))}(0)^\top, F_2(\varphi, \nabla; \Phi, g, B) \mathbf{M}_{(\sigma(1))}(0)^\top, \dots, \mathbf{M}_{(\sigma(\mu'-1))}(0)^\top, \dot{\mathbf{M}}_{(\mu)}(0)^\top]^\top$$

with

$$\begin{aligned}
F_2(\varphi, \nabla; \Phi, g, B) &= -\frac{1}{2} e^{-\varphi^\sharp(\Phi)} \sqrt{-\text{SymDet}(\mathbf{M}(0))}^{-1}, \\
\mathbf{M}_{(\bullet)}(0) &= \text{the } \bullet\text{-th row vector of } \mathbf{M}(0), \\
\mathbf{M}_{\mu\nu}(0) &= \text{the } (\mu, \nu)\text{-entry of } \mathbf{M}(0) = \sum_{i', j'} \varphi^\sharp(E_{i'j'}) D_\mu \varphi^\sharp(y^{i'}) D_\nu \varphi^\sharp(y^{j'}) + 2\pi\alpha' [\nabla_\mu, \nabla_\nu], \\
\dot{\mathbf{M}}_{\mu\nu}(0) &= \sum_{i', j'} R^{E_{i'j'}}[1] |_{\mathbf{y} \mapsto \cdot(\varphi^\sharp(\mathbf{y}d))} \cdot D_\mu \varphi^\sharp(y^{i'}) D_\nu \varphi^\sharp(y^{j'}) \\
&\quad + \sum_{i', j'} \varphi^\sharp(E_{i'j'}) \cdot (D_\mu \varphi^\sharp(y^{i'}) - [\varphi^\sharp(y^{i'}), \dot{A}_\mu]) \cdot D_\nu \varphi^\sharp(y^{j'}) \\
&\quad + \sum_{i', j'} \varphi^\sharp(E_{i'j'}) D_\mu \varphi^\sharp(y^{i'}) \cdot (D_\nu \varphi^\sharp(y^{j'}) - [\varphi^\sharp(y^{j'}), \dot{A}_\nu]) + 2\pi\alpha' (D_\mu \dot{A}_\nu - D_\nu \dot{A}_\mu).
\end{aligned}$$

Remark 5.3.2. [*origin/correction from anomaly equations for open strings*] From the string-theory point of view, it is very important to understand further how such systems of differential equations on the pair (φ, ∇) can arise from or be corrected/improved by the anomaly-free conditions in open-string theory. Cf. Issue (7), Sec. 1.

Remark 5.3.3. [*the case of Hermitian/unitary D-branes*] When in addition E is equipped with a Hermitian structure and φ is Hermitian and ∇ is unitary, the Dirac-Born-Infeld action functional $S_{(\varphi, \nabla)}^{(\Phi, g, B)}$ and, hence, the resulting equations of motion can be simplified. The detail should be studied further. Cf. Remark 2.3.8 and Remark 3.2.5.

6 Remarks on the Chern-Simons/Wess-Zumino term

In view of Polchinski's realization ([Po1]) that a D-brane world-volume can couple to a Ramond-Ramond field in superstring theory (cf. FIGURE 6-0-1), the Chern-Simons/Wess-Zumino term $S_{CS/WZ}$ for D-branes is also an indispensable part to understand the dynamics of D-branes. With the same essence as for the construction of $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla)$, we construct in this section the Chern-Simons/Wess-Zumino action $S_{CS/WZ}(\varphi, \nabla)$ for lower-dimensional D-branes, in which cases anomaly issues do not occur, derive their first variation formula and, hence, obtain their contribution to the equations of motions for D-branes.

To begin, with anomalies taken into account, the coupling of a simple embedded D-brane

$$f : X \hookrightarrow Y$$

with the Ramond-Ramond field C on Y (with a B -field background B), is encoded in the Chern-Simons/Wess-Zumino action for D-branes, which takes the form

$$S_{CS/WZ}^{(C, B)}(f, \nabla) = T_{m-1} \int_X \left(f^* C \wedge e^{2\pi\alpha' F_\nabla + f^* B} \wedge \sqrt{\hat{A}(X)/\hat{A}(N_{X/Y})} \right)_{(m)},$$

where

- $m = \dim X$, T_{m-1} the $D(m-1)$ -brane tension, $\hat{A}(\cdot)$ the \hat{A} -class of the bundle in question, $N_{X/Y}$ the normal bundle of X in Y along f ,
- $(\cdots)_{(m)}$ is the degree- m component of a differential form (\cdots) on X .

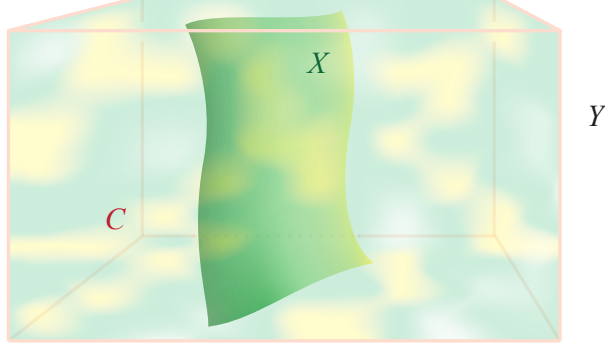


FIGURE 6-0-1. In superstring theory, a D-brane world-volume X can couple to a Ramond-Ramond field C , created by closed superstrings, on the space-time Y . Such coupling influences the dynamics of the D-brane as well. In the FIGURE, the Ramond-Ramond field is indicated by an etherlike foggy background with varying density.

The fact that the over coupling strength is identical with the D-brane tension T_{m-1} is a consequence of supersymmetry. Readers are referred to, e.g. [Bac], [Joh], [Po3:vol. II], [Sz] for more details and references.

With the lesson already learned from studying the Dirac-Born-Infeld action, formally the Chern-Simons/Wess-Zumino action generalizes to the case of coincident D-brane in our setting

$$\varphi : (X^{Az}, E; \nabla) \longrightarrow Y,$$

as

$$S_{CS/WZ}^{(C,B)}(f, \nabla) \stackrel{\text{formally}}{=} T_{m-1} \int_X \text{Re} \left(\text{Tr} \left(\varphi^\diamond C \wedge e^{2\pi\alpha' F_\nabla + \varphi^\diamond B} \wedge \sqrt{\hat{A}(X^{Az})/\hat{A}(N_{X^{Az}/Y})} \right) \right)_{(m)}.$$

One now has to resolve in addition the following issues:

- (8) the anomaly factor “ $\sqrt{\hat{A}(X^{Az})/\hat{A}(N_{X^{Az}/Y})}$ ”, which presumably is an $\text{End}_{\mathbb{C}}(E)$ -valued differential form on X ;
- (9) wedging of of $\text{End}_{\mathbb{C}}(E)$ -valued differential forms on X :

$$\varphi^\diamond C \wedge e^{2\pi\alpha' F_\nabla + \varphi^\diamond B} \wedge \sqrt{\hat{A}(X)/\hat{A}(N_{X^{Az}/Y})}.$$

6.1 Resolution of issues in the Chern-Simons/Wess-Zumino term

We address in this subsection the resolution of Issue (9) in a way that is compatible with how we treat/interpret the Dirac-Born-Infeld action in Sec. 3. This gives us a version of the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C,B)}$ for D-branes of dimension $-1, 0, 1$, and 2 that matches the Dirac-Born-Infeld action $S_{DBI}^{(\Phi,g,B)}$ constructed in Sec. 3.

From determinant function to wedge product of differential forms

For an ordinary differentiable manifold M , the wedge product of differential forms is determined by the wedge product of a collection of 1-forms and the latter is set by the determinant function through the following rule

$$(\omega^1 \wedge \cdots \wedge \omega^s)(e_1 \wedge \cdots \wedge e_s) = \text{Det}(\omega^i(e_j)).$$

Here, e_1, \dots, e_s are vector fields on M , $\omega^1, \dots, \omega^s$ are 1-forms on M , $e_1 \wedge \cdots \wedge e_s := \sum_{\sigma \in \text{Sym}_s} (-1)^\sigma e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(s)}$, and $(\omega^i(e_j))$ is the $s \times s$ matrix with the (i, j) -entry $\omega^i(e_j)$. When $\omega^1, \dots, \omega^s$ are enhanced to 1-forms with value in a noncommutative ring R , the original determinant function $\text{Det}(\cdot)$ needs to be enhanced/generalized as well to a determinant function for matrices with entries in R since now $\omega^j(e_i) \in R$, for $i, j = 1, \dots, s$.

Recall that in the study of non-Abelian Dirac-Born-Infeld action for the pair (φ, ∇) , we ran into the need for such a generalization, too, and introduced the notion of symmetrized determinant SymDet ; cf. Definition 3.1.3.6. There, we propose an Ansatz that this is the determinant function for the construction of the non-Abelian Dirac-Born-Infeld action, cf. Ansatz 3.1.3.11. It is very natural to suggest that the same notion of determinant function is applied to both the Dirac-Born-Infeld term and the Chern/Simons/Wess-Zumino term in the full action for D-branes:

Ansatz 6.1.1. [wedge product in the Chern-Simons/Wess-Zumino action] We interpret the wedge products that appear in the formal expression for the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C,B)}$ through the symmetrized determinant that applies to the above defining identities for wedge product; namely, we require that

$$(\omega^1 \wedge \cdots \wedge \omega^s)(e_1 \wedge \cdots \wedge e_s) = \text{SymDet}(\omega^i(e_j))$$

for $\text{End}_{\mathbb{C}}(E)$ -valued 1-forms $\omega^1, \dots, \omega^s$ on X . Denote this generalized wedge product by $\overset{\circ}{\wedge}$.

Example 6.1.2. $[C_{(1)} \overset{\circ}{\wedge} F \overset{\circ}{\wedge} F]$ Let $C_{(1)} = \sum_{\mu} C_{\mu} dx^{\mu}$ and $F = \sum_{\mu', \nu'} F_{\mu' \nu'} dx^{\mu'} \wedge dx^{\nu'}$ be an $\text{End}_{\mathbb{C}}(E)$ -valued 1-form and 2-form respectively, then

$$C_{(1)} \overset{\circ}{\wedge} F \overset{\circ}{\wedge} F = \sum_{\mu, \mu', \nu', \mu'', \nu''} (C_{\mu} \odot F_{\mu' \nu'} \odot F_{\mu'' \nu''}) dx^{\mu} \wedge dx^{\mu'} \wedge dx^{\nu'} \wedge dx^{\mu''} \wedge dx^{\nu''},$$

where, recall that, $C_{\mu} \odot F_{\mu' \nu'} \odot F_{\mu'' \nu''}$ is the symmetrized product of the triple $(C_{\mu}, F_{\mu' \nu'}, F_{\mu'' \nu''})$.

Remark 6.1.3. [on the ring $(C^{\infty}(\bigwedge^{\bullet} T^*X \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)), +, \overset{\circ}{\wedge})$] (Cf. Remark 3.1.3.10.) Properties of $\overset{\circ}{\wedge}$ follow from properties of \odot on $C^{\infty}(\text{End}_{\mathbb{C}}(E))$ and properties of \wedge on $C^{\infty}(\bigwedge^{\bullet} T^*X)$. In particular, for example, $C_{(1)} \overset{\circ}{\wedge} F \overset{\circ}{\wedge} F$ is directly defined for the triple $(C_{(1)}, F, F)$ of $\text{End}_{\mathbb{C}}(E)$ -valued differential forms on X , rather than through a train of applications of a binary operation. The three elements in $\bigwedge^5 T^*X \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)$

$$C_{(1)} \overset{\circ}{\wedge} F \overset{\circ}{\wedge} F, \quad (C_{(1)} \overset{\circ}{\wedge} F) \overset{\circ}{\wedge} F, \quad C_{(1)} \overset{\circ}{\wedge} (F \overset{\circ}{\wedge} F)$$

in general are all different. The ring $(C^{\infty}(\bigwedge^{\bullet} T^*X \otimes_{\mathbb{R}} \text{End}_{\mathbb{C}}(E)), +, \overset{\circ}{\wedge})$ is \mathbb{Z}_2 -graded, \mathbb{Z}_2 -commutative, but not associative.

Lemma 6.1.4. $[\varphi^\diamond, \wedge, \text{ and } \overset{\circ}{\wedge}]$ Let $\varphi : (X^{Az}, E, \nabla) \rightarrow Y$ be an admissible map and ζ_1, \dots, ζ_k differential forms on Y . Then

$$\varphi^\diamond \zeta_1 \overset{\circ}{\wedge} \dots \overset{\circ}{\wedge} \varphi^\diamond \zeta_k = \varphi^\diamond (\zeta_1 \wedge \dots \wedge \zeta_k).$$

Proof. Recall the surrogate X_φ of X^{Az} specified by φ and the built-in maps

$$\begin{array}{ccc} X_\varphi & \xrightarrow{f_\varphi} & Y \\ \downarrow \pi_\varphi & & \\ X & & \end{array}.$$

Since the function-ring $A_\varphi := C^\infty(X) \langle \text{Im } \varphi^\sharp \rangle$ of X_φ is commutative, for differential forms $\zeta'_1, \dots, \zeta'_k$ on X_φ ,

$$\pi_{\varphi*} \zeta'_1 \overset{\circ}{\wedge} \dots \overset{\circ}{\wedge} \pi_{\varphi*} \zeta'_k = \pi_{\varphi*} \zeta'_1 \wedge \dots \wedge \pi_{\varphi*} \zeta'_k = \pi_{\varphi*} (\zeta'_1 \wedge \dots \wedge \zeta'_k).$$

It follows that

$$\begin{aligned} \varphi^\diamond \zeta_1 \overset{\circ}{\wedge} \dots \overset{\circ}{\wedge} \varphi^\diamond \zeta_k &= \pi_{\varphi*} (f_\varphi^* \zeta_1) \overset{\circ}{\wedge} \dots \overset{\circ}{\wedge} \pi_{\varphi*} (f_\varphi^* \zeta_k) \\ &= \pi_{\varphi*} (f_\varphi^* \zeta_1 \wedge \dots \wedge f_\varphi^* \zeta_k) = \pi_{\varphi*} (f_\varphi^* (\zeta_1 \wedge \dots \wedge \zeta_k)) = \varphi^\diamond (\zeta_1 \wedge \dots \wedge \zeta_k). \end{aligned}$$

□

The Chern-Simons/Wess-Zumino action for lower dimensional D-branes

For a simple D-brane world-volume $f : X \hookrightarrow Y$, the anomaly factor $\sqrt{\hat{A}(X)/\hat{A}(N_{X/Y})} = 1$, for $\dim X = m \leq 3$. This may not hold for φ since $\varphi(X^{Az})$ can have fuzzy/nilpotent structure of nilpotency $\leq r$ (the rank of E as a complex vector bundle on X), which can be large even when the dimension m of X is small. However, if one formally assume that the same is true, then for lower dimensional D-branes (i.e. D(-1)-, D0-, D1-, D2-branes), one has: (Assuming that $B = \sum_{i,j} B_{ij} dy^i \otimes dy^j$, $B_{ji} = -B_{ij}$)

- For $D(-1)$ -brane world-point ($m = 0$):

$$S_{CS/WZ}^{(C_{(0)})}(\varphi) = T_{-1} \cdot \text{Tr}(\varphi^\diamond C_{(0)}) = T_{-1} \cdot \text{Tr}(\varphi^\sharp(C_{(0)})).$$

- For D -particle world-line ($m = 1$): Assume that $C_{(1)} = \sum_{i=1}^n C_i dy^i$ locally; then

$$S_{CS/WZ}^{(C_{(1)})}(\varphi, \nabla) = T_0 \int_X \text{Tr}(\varphi^\diamond C_{(1)}) \stackrel{\text{locally}}{=} T_0 \int_U \text{Tr} \left(\sum_{i=1}^n \varphi^\sharp(C_i) \cdot D_x \varphi^\sharp(y^i) \right) dx.$$

Here, $D_x := D_{\partial/\partial x}$.

- For *D-string* world-sheet ($m = 2$): Assume that $C_{(2)} = \sum_{i,j=1}^n C_{ij} dy^i \otimes dy^j$ locally, with $C_{ij} = -C_{ji}$; then

$$\begin{aligned}
S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}(\varphi, \nabla) &= T_1 \int_X \text{Re}(\text{Tr}(\varphi^\diamond C_{(2)} + \varphi^\diamond(C_{(0)}B) + 2\pi\alpha' \varphi^\sharp(C_{(0)}) \odot F_\nabla)) \\
&= T_1 \int_X \text{Re}(\text{Tr}(\varphi^\diamond(C_{(2)} + C_{(0)}B) + \pi\alpha' \varphi^\sharp(C_{(0)})F_\nabla + \pi\alpha' F_\nabla \varphi^\sharp(C_{(0)}))) \\
&\stackrel{\text{locally}}{=} T_1 \int_U \text{Re}\left(\text{Tr}\left(\sum_{i,j=1}^n \varphi^\sharp(C_{ij} + C_{(0)}B_{ij}) D_{x^1} \varphi^\sharp(y^i) D_{x^2} \varphi^\sharp(y^j) \right.\right. \\
&\quad \left.\left. + \pi\alpha' \varphi^\sharp(C_{(0)}) [\nabla_{x^1}, \nabla_{x^2}] + \pi\alpha' [\nabla_{x^1}, \nabla_{x^2}] \varphi^\sharp(C_{(0)})\right)\right) d^2 \mathbf{x}.
\end{aligned}$$

Here, $D_{x^1} := D_{\partial/\partial x^1}$, $D_{x^2} := D_{\partial/\partial x^2}$ and $\nabla_{x^1} := \nabla_{\partial/\partial x^1}$, $\nabla_{x^2} := \nabla_{\partial/\partial x^2}$.

- For *D-membrane* world-volume ($m = 3$): Assume that $C_{(1)} = \sum_{i=1}^n C_i dy^i$ and $C_{(3)} = \sum_{i,j,k=1}^n C_{ijk} dy^i \otimes dy^j \otimes dy^k$ locally, with C_{ijk} alternating with respect to ijk ; then

$$\begin{aligned}
S_{CS/WZ}^{(C_{(1)}, C_{(3)}, B)}(\varphi, \nabla) &= T_2 \int_X \text{Re}(\text{Tr}(\varphi^\diamond C_{(3)} + \varphi^\diamond(C_{(1)} \wedge B) + 2\pi\alpha' \varphi^\diamond C_{(1)} \overset{\circ}{\wedge} F_\nabla)) \\
&\stackrel{\text{locally}}{=} T_2 \int_U \text{Re}\left(\text{Tr}\left(\sum_{i,j,k=1}^n \varphi^\sharp(C_{ijk} + C_i B_{jk} + C_j B_{ki} + C_k B_{ij}) D_{x^1} \varphi^\sharp(y^i) D_{x^2} \varphi^\sharp(y^j) D_{x^3} \varphi^\sharp(y^k) \right.\right. \\
&\quad \left.\left. + \pi\alpha' \sum_{(\lambda\mu\nu) \in \text{Sym}_3} \sum_{i=1}^n (-1)^{(\lambda\mu\nu)} \left(\varphi^\sharp(C_i) D_{x^\lambda} (\varphi^\sharp(y^i)) [\nabla_{x^\mu}, \nabla_{x^\nu}] + [\nabla_{x^\mu}, \nabla_{x^\nu}] \varphi^\sharp(C_i) D_{x^\lambda} \varphi^\sharp(y^i) \right) \right)\right) d^3 \mathbf{x}.
\end{aligned}$$

The technical issue of anomaly is the focus of another work. For the moment, we will take the above as our *working Ansatz* for the Chern-Simons/Wess-Zumino action for lower-dimensional D-branes.

Remark 6.1.5. [What is Ramond-Ramond field?] From the way a D-brane couple to a Ramond-Ramond field, one learns that a Ramond-Ramond field is to a D-brane as a B -field is to a fundamental string. In the latter case, while a B -field is taken to be a 2-form on the space-time Y to begin with, after years of development one learns that the meaning/precise definition of B -field goes much beyond just a 2-form on Y . It's not yet settled what it really is, but it is known that structures on loop spaces and gerbes are involved (e.g. [Bry]). One expects thus that, in parallel, a Ramond-Ramond field go beyond just a differential form on the space-time Y . Under our setting, the loop space in the case of B -field is expected to be replaced by a map-space $\text{Map}((Z^{\text{Az}}, E; \nabla), Y)$, where Z^{Az} is an Azumaya/matrix manifold representing a D-brane (not D-brane world-volume). For example, the Ramond-Ramond 2-field $C_{(2)}$ in the Type IIB superstring theory, when fully developed, is expected to be related to a matrix-loop space $\text{Map}((S^{1, \text{Az}}, E, \nabla), Y)$ and structures thereupon. Furthermore, when E has rank > 1 , one expects also that $C_{(2)}$, being a field sourced by D-strings, is enhanced to non-Abelian-valued. All these issues, and beyond, remain to be understood.

6.2 The first variation and the contribution to the equations of motion

Under the same setup as in Sec. 5.2, we derive in this subsection the first variation of the Chern-Simons/Wess-Zumino action $S_{CS/WZ}^{(C, B)}$ for lower-dimensional D-brane world-volumes. The additional contribution to the equations of motion for such lower-dimensional D-branes due to the additional term $S_{CS/WZ}^{(C, B)}$ in the total action for D-brane world-volume would then follow.

6.2.1 D(-1)-brane world-point ($m = 0$)

For a D(-1)-brane world-point, $\dim X = 0$, $\nabla = 0$, and $S_{CS/WZ}^{(C_{(0)})}(\varphi) = T_{-1} \cdot \text{Tr}(\varphi^\sharp(C_{(0)}))$. It follows that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} S_{CS/WZ}^{(C_{(0)})}(\varphi_T) &= T_{-1} \left. \frac{d}{dt} \right|_{t=0} \text{Tr}(\varphi_T^\sharp(C_{(0)})) = T_{-1} \text{Tr} \left(\left. \frac{d}{dt} \right|_{t=0} \varphi_T^\sharp(C_{(0)}) \right) \\ &= T_{-1} \text{Tr} \left(\sum_{j=1}^n \left(\sum_{\mathbf{d}, \bar{\mathbf{d}}, \bar{\pi}} R^{C_{(0)}}[1]_{(\bar{\mathbf{d}}, \bar{\pi})}^R(\varphi^\sharp(\mathbf{y})) \cdot R^{C_{(0)}}[1]_{(\bar{\mathbf{d}}, \bar{\pi})}^L(\varphi^\sharp(\mathbf{y})) \right) \cdot \dot{\varphi}^\sharp(y^j) \right) \\ &=: T_{-1} \text{Tr} \left(\sum_{j=1}^n \mathcal{NL}_j^{(C_{(0)}); \delta\varphi}(\varphi) \cdot \dot{\varphi}^\sharp(y^j) \right). \end{aligned}$$

Here, the following identities are employed:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \left(\varphi_T^\sharp(C_{(0)}) \right) &= R^{C_{(0)}}[1]_{\mathbf{y}^{\mathbf{d}} \rightsquigarrow (\varphi^\sharp(\mathbf{y}^{\mathbf{d}}))} \\ &= \sum_{j=1}^n \sum_{\mathbf{d}=0}^{\bullet} \sum_{\mathbf{d}, |\mathbf{d}|=d} \sum_{\bar{\pi} \in \vec{P}tn(1, d), i_{(\bar{\pi}, \mathbf{d})}=j} \sum \\ &\quad ([\partial_{y^j}^{\bar{\pi}}] R^{C_{(0)}}[1]_{(\mathbf{d})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^j) \cdot ([\partial_{y^j}^{\bar{\pi}}] R^{C_{(0)}}[1]_{(\mathbf{d})}^R(\varphi^\sharp(\mathbf{y}))) \\ &=: \sum_{j=1}^n \sum_{\mathbf{d}, \bar{\mathbf{d}}, \bar{\pi}} R^{C_{(0)}}[1]_{(\bar{\mathbf{d}}, \bar{\pi})}^L(\varphi^\sharp(\mathbf{y})) \cdot \dot{\varphi}^\sharp(y^j) \cdot R^{C_{(0)}}[1]_{(\bar{\mathbf{d}}, \bar{\pi})}^R(\varphi^\sharp(\mathbf{y})). \end{aligned}$$

In this case, $S_{DBI}^{(\Phi, g, B)}(\varphi) = 0$ always and the full action $S_{DBI}^{(\Phi, g, B)} + S_{CS/WZ}^{(C_{(0)})}$ is simply $S_{CS/WZ}^{(C_{(0)})}$. The full system of equations of motion is thus

$$\mathcal{NL}_j^{(\Phi, g, B, C_{(0)}); \delta\varphi}(\varphi) := \mathcal{NL}_j^{(C_{(0)}); \delta\varphi}(\varphi) = 0,$$

$j = 1, \dots, n$, for D(-1)-brane. Such world-points give rise to instantons in space-time.

6.2.2 D-particle world-line ($m = 1$)

For a D-particle world-line, $\dim X = 1$ and

$$S_{CS/WZ}^{(C_{(1)})}(\varphi, \nabla) = T_0 \int_U \text{Tr} \left(\sum_{i=1}^n \varphi^\sharp(C_i) \cdot D_x \varphi^\sharp(y^i) \right) dx$$

locally over X . It follows that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} S_{CS/WZ}^{(C_{(1)})}(\varphi_T, \nabla^T) &= T_0 \int_U \text{Tr} \left(\sum_{i=1}^n \dot{\varphi}^\sharp(C_i) \cdot D_x \varphi^\sharp(y^i) + \varphi^\sharp(C_i) \cdot (D_x \dot{\varphi}^\sharp(y^i) - [\varphi^i(\sharp)(y^i), \dot{A}_x]) \right) dx \\ &= T_0 \text{Tr} \left(\sum_{i=1}^n \varphi^\sharp(C_i) \dot{\varphi}^\sharp(y^i) \right) \Big|_{\partial U} \\ &\quad + T_0 \int_U \text{Tr} \left(\sum_{i=1}^n \varphi^\sharp(C_i) \cdot D_x \varphi^\sharp(y^i) - D_x \varphi^\sharp(C_i) \cdot \dot{\varphi}^\sharp(y^i) - \varphi^\sharp(C_i) \cdot [\varphi^i(\sharp)(y^i), \dot{A}_x] \right) dx \\ &= T_0 \text{Tr} \left(\varphi^{(C_{(1)})}(\dot{\varphi}^\sharp(\mathbf{y})) \right) \Big|_{\partial U} + T_0 \int_U \text{Tr} \left(\sum_{j=1}^n \mathcal{NL}_j^{(C_{(1)}); \delta\varphi}(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^j) + \mathcal{NL}_x^{(C_{(1)}); \delta\nabla}(\varphi) \cdot \dot{A}_x \right) dx, \end{aligned}$$

where

$$\begin{aligned}
B\Gamma^{(\varphi; C_{(1)})}(\dot{\varphi}^\sharp(\mathbf{y})) &= \text{Tr} \left(\sum_{i=1}^n \varphi^\sharp(C_i) \dot{\varphi}^\sharp(y^i) \right), \\
\mathcal{NL}_j^{(C_{(1)}); \delta\varphi}(\varphi, \nabla) &= -D_x \varphi^\sharp(C_j) + \sum_{i=1}^n \sum_{d, \mathbf{d}, \pi; |\mathbf{d}|=d, i_{(\pi, \mathbf{d})}=j} R^{C_i}[1]_{(\mathbf{d}, \pi)}^R(\varphi^\sharp(\mathbf{y})) \cdot D_x \varphi^\sharp(y^i) \cdot R^{C_i}[1]_{(\mathbf{d}, \pi)}^L(\varphi^\sharp(\mathbf{y})), \\
\mathcal{NL}_x^{(C_{(1)}); \delta\nabla}(\varphi) &= \sum_{i=1}^n [\varphi^\sharp(y^i), \varphi^\sharp(C_i)] = 0.
\end{aligned}$$

The full action $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla) + S_{CS/WZ}^{(C_{(1)})}(\varphi, \nabla)$ gives the system of equations of motion for a D-particle moving in Y :

$$\begin{aligned}
\mathcal{NL}_j^{(\Phi, g, B, C_{(1)}); \delta\varphi}(\varphi, \nabla) &:= \mathcal{NL}_j^{(\Phi, g, B); \delta\varphi}(\varphi, \nabla) + \mathcal{NL}_j^{(C_{(1)}); \delta\varphi}(\varphi, \nabla) = 0, \\
\mathcal{NL}_x^{(\Phi, g, B, C_{(1)}); \delta\nabla}(\varphi, \nabla) &:= \mathcal{NL}_x^{(\Phi, g, B); \delta\nabla}(\varphi, \nabla) = 0,
\end{aligned}$$

$j = 1, \dots, n$.

For the current case, the curvature F_∇ of ∇ is zero and the above system may still involves A_x but not its differentials with respect to x . I.e. it is a system of differential equations on φ but non-differential equations on ∇ . ∇ is thus non-dynamical, as is anticipated. Thus, after a re-trivialization for the fundamental module E on X , one may assume that $A_x \equiv 0$ and the above system is reduced to a system

$$\mathcal{NL}^{(\Phi, g, B, C_{(1)}); \delta\varphi}(\varphi) = 0, \quad j = 1, \dots, n,$$

of second-order nonlinear differential equations that involve φ alone.

6.2.3 D-string world-sheet ($m = 2$)

Denote

$$\check{C}_{(2)} := C_{(2)} + C_{(0)}B = \sum_{ij} (C_{ij} + C_{(0)}B_{ij}) dy^i \otimes dy^j = \sum_{i,j} \check{C}_{ij} dy^i \otimes dy^j$$

in local coordinates of Y . Then, for a D-string world-sheet, $\dim X = 2$ and

$$\begin{aligned}
S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}(\varphi, \nabla) &= T_1 \int_U \text{Re} \left(\text{Tr} \left(\sum_{i,j=1}^n \varphi^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j) \right. \right. \\
&\quad \left. \left. + \pi \alpha' \varphi^\sharp(C_{(0)}) F_{12} + \pi \alpha' F_{12} \varphi^\sharp(C_{(0)}) \right) \right) d^2 \mathbf{x}
\end{aligned}$$

locally over X . (Here, $D_1 := D_{\partial/\partial x^1}$, $D_2 := D_{\partial/\partial x^2}$, and $F_{12} := [\nabla_{x^1}, \nabla_{x^2}]$ is the curvature of ∇ .) It follows then from a straightforward computation that

$$\begin{aligned}
&\left. \frac{d}{dt} \right|_{t=0} S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}(\varphi_T, \nabla^T) \\
&= T_1 \int_U \text{Re} \left(\text{Tr} \left(\sum_{i,j=1}^n \left(\dot{\varphi}^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j) + \varphi^\sharp(\check{C}_{ij}) (D_1 \dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_1]) D_2 \varphi^\sharp(y^j) \right. \right. \right. \\
&\quad \left. \left. + \varphi^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) (D_2 \dot{\varphi}^\sharp(y^j) - [\varphi^\sharp(y^j), \dot{A}_2]) \right) \right. \\
&\quad \left. + \pi \alpha' \dot{\varphi}^\sharp(C_{(0)}) F_{12} + \pi \alpha' \varphi^\sharp(C_{(0)}) (D_1 \dot{A}_2 - D_2 \dot{A}_1) \right. \\
&\quad \left. + \pi \alpha' (D_1 \dot{A}_2 - D_2 \dot{A}_1) \varphi^\sharp(C_{(0)}) + \pi \alpha' F_{12} \dot{\varphi}^\sharp(C_{(0)}) \right) d^2 \mathbf{x} \\
&= T_1 \int_{\partial U} \text{Re} (B\Gamma^{(\varphi, \nabla; C_{(0)}, C_{(2)}, B)}(\dot{\varphi}^\sharp(\mathbf{y}), \dot{\mathbf{A}})) \\
&\quad + T_1 \int_U \text{Re} \left(\text{Tr} \left(\sum_{j=1}^n \mathcal{NL}_j^{(C_{(0)}, C_{(2)}, B); \delta\varphi}(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^j) + \sum_{\nu=1}^2 \mathcal{NL}_\nu^{(C_{(0)}); \delta\nabla}(\varphi, \nabla) \cdot \dot{A}_\nu \right) \right) d^2 \mathbf{x},
\end{aligned}$$

where

- the *boundary term* is given by

$$\begin{aligned} & B\Gamma^{(\varphi, \nabla; C_{(0)}, C_{(2)}, B)}(\dot{\varphi}^\sharp(\mathbf{y}), \dot{\mathbf{A}}), \quad \text{a 1-form on } U, \\ & = \text{Tr} \left(\sum_{j=1}^n \left(\sum_{i=1}^n D_2 \varphi^\sharp(y^i) \varphi^\sharp(\check{C}_{ji}) \right) \cdot \dot{\varphi}^\sharp(y^j) + 2\pi\alpha' \varphi^\sharp(C_{(0)}) \cdot \dot{A}_2 \right) dx^2 \\ & \quad - \text{Tr} \left(\sum_{j=1}^n \left(\sum_{i=1}^n \varphi^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) \right) \cdot \dot{\varphi}^\sharp(y^j) - 2\pi\alpha' \varphi^\sharp(C_{(0)}) \cdot \dot{A}_1 \right) dx^1, \end{aligned}$$

- the *subsystem associated to variations of φ* :

$$\begin{aligned} & \mathcal{NL}_j^{(C_{(0)}, C_{(2)}, B); \delta\varphi}(\varphi, \nabla) \\ & = \sum_{i', j'=1}^n \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i_{(\vec{\pi}, \mathbf{d})}=j} R^{\check{C}_{i'j'}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})) D_1 \varphi^\sharp(y^{i'}) D_2 \varphi^\sharp(y^{j'}) R^{\check{C}_{i'j'}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})) \\ & \quad - \sum_{i=1}^n \left(D_2 \varphi^\sharp(y^i) D_1 \varphi^\sharp(\check{C}_{ji}) + D_2 \varphi^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) + [F_{12}, \varphi^\sharp(y^i)] \cdot \varphi^\sharp(\check{C}_{ji}) \right) \\ & \quad + 2\pi\alpha' \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i_{(\vec{\pi}, \mathbf{d})}=j} R^{C_{(0)}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})) F_{12} R^{C_{(0)}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})), \end{aligned}$$

- the *subsystem associated to variations of ∇* :

$$\mathcal{NL}_1^{(C_{(0)}); \delta\nabla}(\varphi, \nabla) = 2\pi\alpha' D_2 \varphi^\sharp(C_{(0)}), \quad \mathcal{NL}_2^{(C_{(0)}); \delta\nabla}(\varphi, \nabla) = -2\pi\alpha' D_1 \varphi^\sharp(C_{(0)}).$$

Note that, as a consequence of Leibniz rule or integration by parts, there are at first summands

$$\begin{aligned} & -D_2 \varphi^\sharp(y^j) \varphi^\sharp(\check{C}_{ij}) \varphi^\sharp(y^i) + \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j) \varphi^\sharp(\check{C}_{ij}) \quad \text{in } \mathcal{NL}_1^{(C_{(0)}); \delta\nabla}(\varphi, \nabla), \\ & -\varphi^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) \varphi^\sharp(y^j) + \varphi^\sharp(y^j) \varphi^\sharp(\check{C}_{ij}) D_1 \varphi^\sharp(y^i) \quad \text{in } \mathcal{NL}_2^{(C_{(0)}); \delta\nabla}(\varphi, \nabla), \end{aligned}$$

respectively. However, they vanish for (φ, ∇) admissible. Thus, the 2-forms $C_{(2)}$ and B has no consequence to the variation of $S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}$ with respect to ∇ . This is anticipated since there is no coupling term between $C_{(2)}$, B and ∇ in $S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}$.

The contribution of the Chern-Simon/Wess-Zumino term $S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}$ to the equations of motion for a D -string follows immediately.

6.2.4 D-membrane world-volume ($m = 3$)

Denote

$$\begin{aligned} \check{C}_{(3)} & := C_{(3)} + C_{(1)} \wedge B \\ & = \sum_{i,j,k} (C_{ijk} + C_i B_{jk} + C_j B_{ki} + C_k B_{ij}) dy^i \otimes dy^j \otimes dy^k = \sum_{i,j,k} \check{C}_{ijk} dy^i \otimes dy^j \otimes dy^k \end{aligned}$$

in local coordinates of Y . Then, for D-membrane world-volume, $\dim X = 3$ and

$$\begin{aligned} & S_{CS/WZ}^{(C_{(1)}, C_{(3)}, B)}(\varphi, \nabla) \\ &= T_2 \int_U \text{Re} \left(\text{Tr} \left(\sum_{i,j,k=1}^n \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j) D_3 \varphi^\sharp(y^k) \right. \right. \\ & \quad \left. \left. + 2\pi\alpha' \sum_{(\lambda\mu\nu) \in \text{Sym}_3} \sum_{i=1}^n (-1)^{(\lambda\mu\nu)} \left(\varphi^\sharp(C_i) D_\lambda \varphi^\sharp(y^i) F_{\mu\nu} \right) \right) \right) d^3 \mathbf{x} \end{aligned}$$

locally over X . (Here, $F_{\mu\nu} := [\nabla_{x^\mu}, \nabla_{x^\nu}]$ is the curvature of ∇ .) It follows then from a straightforward computation that

$$\begin{aligned} & \left. \frac{d}{dt} \right|_{t=0} S_{CS/WZ}^{(C_{(1)}, C_{(3)}, B)}(\varphi_T, \nabla^T) \\ &= T_2 \int_U \text{Re} \left(\text{Tr} \left(\sum_{i,j,k=1}^n \left(\dot{\varphi}^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j) D_3 \varphi^\sharp(y^k) \right. \right. \right. \\ & \quad \left. \left. + \varphi^\sharp(\check{C}_{ijk}) \cdot (D_1 \dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_1]) \cdot D_2 \varphi^\sharp(y^j) D_3 \varphi^\sharp(y^k) \right. \right. \\ & \quad \left. \left. + \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) \cdot (D_2 \dot{\varphi}^\sharp(y^j) - [\varphi^\sharp(y^j), \dot{A}_2]) \cdot D_3 \varphi^\sharp(y^k) \right. \right. \\ & \quad \left. \left. + \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j) \cdot (D_3 \dot{\varphi}^\sharp(y^k) - [\varphi^\sharp(y^k), \dot{A}_3]) \right) \right. \\ & \quad \left. + 2\pi\alpha' \sum_{(\lambda\mu\nu) \in \text{Sym}_3} \sum_{i=1}^n (-1)^{(\lambda\mu\nu)} \left(\dot{\varphi}^\sharp(C_i) D_\lambda \varphi^\sharp(y^i) F_{\mu\nu} \right. \right. \\ & \quad \left. \left. + \varphi^\sharp(C_i) \cdot (D_\lambda \dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_\lambda]) \cdot F_{\mu\nu} + \varphi^\sharp(C_i) D_\lambda \varphi^\sharp(y^i) \cdot (D_\mu \dot{A}_\nu - D_\nu \dot{A}_\mu) \right) \right) d^3 \mathbf{x} \\ &= T_2 \int_{\partial U} \text{Re} (B\Gamma^{(\varphi, \nabla; C_{(1)}, C_{(3)}, B)}(\dot{\varphi}^\sharp(\mathbf{y}), \dot{\mathbf{A}})) \\ & \quad + T_2 \int_U \text{Re} \left(\text{Tr} \left(\sum_{j=1}^n N_j^{(C_{(1)}, C_{(3)}, B); \delta\varphi}(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^j) + \sum_{\nu=1}^3 N_\nu^{(C_{(1)}) ; \delta\nabla}(\varphi, \nabla) \cdot \dot{A}_\nu \right) \right) d^3 \mathbf{x}, \end{aligned}$$

where

• the *boundary term* is given by

$$\begin{aligned} & B\Gamma^{(\varphi, \nabla; C_{(1)}, C_{(3)}, B)}(\dot{\varphi}^\sharp(\mathbf{y}), \dot{\mathbf{A}}), \quad \text{a 2-form on } U, \\ &= \text{Tr} \left(\sum_{j=1}^n \left(\sum_{i,k=1}^n D_2 \varphi^\sharp(y^i) D_3 \varphi^\sharp(y^k) \varphi^\sharp(\check{C}_{jik}) + 4\pi\alpha' F_{23} \varphi^\sharp(C_j) \right) \cdot \dot{\varphi}^\sharp(y^j) \right. \\ & \quad \left. + 4\pi\alpha' \left(\sum_{i=1}^n \varphi^\sharp(C_i) D_3 \varphi^\sharp(y^i) \right) \cdot \dot{A}_2 - 4\pi\alpha' \left(\sum_{i=1}^n \varphi^\sharp(C_i) D_2 \varphi^\sharp(y^i) \right) \cdot \dot{A}_3 \right) d^2 \wedge dx^3 \\ & - \text{Tr} \left(\sum_{j=1}^n \left(\sum_{i,k=1}^n D_3 \varphi^\sharp(y^k) \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) - 4\pi\alpha' F_{13} \varphi^\sharp(C_j) \right) \cdot \dot{\varphi}^\sharp(y^j) \right. \\ & \quad \left. - 4\pi\alpha' \left(\sum_{i=1}^n \varphi^\sharp(C_i) D_3 \varphi^\sharp(y^i) \right) \cdot \dot{A}_1 + 4\pi\alpha' \left(\sum_{i=1}^n \varphi^\sharp(C_i) D_1 \varphi^\sharp(y^i) \right) \cdot \dot{A}_3 \right) d^1 \wedge dx^3 \\ & + \text{Tr} \left(\sum_{j=1}^n \left(\sum_{i,k=1}^n \varphi^\sharp(\check{C}_{ikj}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^k) + 4\pi\alpha' F_{12} \varphi^\sharp(C_j) \right) \cdot \dot{\varphi}^\sharp(y^j) \right. \\ & \quad \left. + 4\pi\alpha' \left(\sum_{i=1}^n \varphi^\sharp(C_i) D_2 \varphi^\sharp(y^i) \right) \cdot \dot{A}_1 - 4\pi\alpha' \left(\sum_{i=1}^n \varphi^{C_i} D_1 \varphi^\sharp(y^i) \right) \cdot \dot{A}_2 \right) dx^1 \wedge dx^2, \end{aligned}$$

- the *subsystem associated to variations of φ* :

$$\begin{aligned}
& \mathcal{N}_j^{(C_{(1)}, C_{(3)}, B); \delta\varphi}(\varphi, \nabla) \\
&= \sum_{i', j', k'=1}^n \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i(\vec{\pi}, \mathbf{d})=j} R^{\check{C}_{i'j'k'}}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})) D_1 \varphi^\sharp(y^{i'}) D_2 \varphi^\sharp(y^{j'}) D_3 \varphi^\sharp(y^{k'}) R^{\check{C}_{i'j'k'}}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})) \\
&\quad - \sum_{i,k=1}^n \left(D_2 \varphi^\sharp(y^i) D_3 \varphi^\sharp(y^k) D_1 \varphi^\sharp(\check{C}_{jik}) + D_3 \varphi^\sharp(y^k) D_2 \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) \right. \\
&\quad \quad \quad + D_3 \varphi^\sharp(\check{C}_{ikj}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^k) + [F_{12}, \varphi^\sharp(y^i)] \cdot D_3 \varphi^\sharp(y^k) \varphi^\sharp(\check{C}_{jik}) \\
&\quad \quad \quad \left. + [F_{23}, \varphi^\sharp(y^k)] \cdot \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) + [F_{31}, \varphi^\sharp(y^i)] \cdot D_2 \varphi^\sharp(y^k) \varphi^\sharp(\check{C}_{ikj}) \right) \\
&\quad + 2\pi\alpha' \sum_{i=1}^n \sum_{(\lambda\mu\nu) \in Sym_3} \sum_{d, \mathbf{d}, \vec{\pi}; |\mathbf{d}|=d, i(\vec{\pi}, \mathbf{d})=j} (-1)^{(\lambda\mu\nu)} R^{C_i}[1]_{(\mathbf{d}, \vec{\pi})}^R(\varphi^\sharp(\mathbf{y})) D_\lambda \varphi^\sharp(y^i) F_{\mu\nu} R^{C_i}[1]_{(\mathbf{d}, \vec{\pi})}^L(\varphi^\sharp(\mathbf{y})) \\
&\quad - 2\pi\alpha' \sum_{(\lambda\mu\nu) \in Sym_3} (-1)^{(\lambda\mu\nu)} \left(F_{\mu\nu} D_\lambda \varphi^\sharp(C_j) + D_\lambda F_{\mu\nu} \varphi^\sharp(C_j) \right),
\end{aligned}$$

- the *subsystem associated to variations of ∇* :

$$\begin{aligned}
& \mathcal{N}_\lambda^{(C_{(1)}); \delta\nabla}(\varphi, \nabla) \\
&= 2\pi\alpha' \sum_{i=1}^n [\varphi^\sharp(y^i), F_{\mu\nu} \varphi^\sharp(C_i)] \\
&\quad + 4\pi\alpha' \sum_{i=1}^n \left(D_\mu \varphi^\sharp(C_i) D_\nu \varphi^\sharp(y^i) - D_\nu \varphi^\sharp(C_i) D_\mu \varphi^\sharp(y^i) + \varphi^\sharp(C_i) \cdot [F_{\mu\nu}, \varphi^\sharp(y^i)] \right),
\end{aligned}$$

where $(\lambda\mu\nu) = (123), (231), (312)$.

Note that, as a consequence of Leibniz rule or integration by parts, there are at first summands

$$\begin{aligned}
& \sum_{i,j,k=1}^n [\varphi^\sharp(y^i), D_2 \varphi^\sharp(y^j) D_3 \varphi^\sharp(y^k) \varphi^\sharp(\check{C}_{ijk})] \quad \text{in } \mathcal{N}_1^{(C_{(1)}); \delta\nabla}(\varphi, \nabla), \\
& \sum_{i,j,k=1}^n [\varphi^\sharp(y^j), D_3 \varphi^\sharp(y^k) \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i)] \quad \text{in } \mathcal{N}_2^{(C_{(1)}); \delta\nabla}(\varphi, \nabla), \\
& \sum_{i,j,k=1}^n [\varphi^\sharp(y^k), \varphi^\sharp(\check{C}_{ijk}) D_1 \varphi^\sharp(y^i) D_2 \varphi^\sharp(y^j)] \quad \text{in } \mathcal{N}_3^{(C_{(1)}); \delta\nabla}(\varphi, \nabla)
\end{aligned}$$

respectively. However, they vanish for (φ, ∇) admissible. Thus, the 3-forms $C_{(3)}$ and $C_{(1)} \wedge B$ have no consequence to the variation of $S_{CS/WZ}^{(C_{(1)}, C_{(3)}, B)}$ with respect to ∇ . This is anticipated since there is no coupling term between $C_{(3)}$, $C_{(1)} \wedge B$ and ∇ in $S_{CS/WZ}^{(C_{(1)}, C_{(3)}, B)}$.

The contribution of the Chern-Simon/Wess-Zumino term $S_{CS/WZ}^{(C_{(1)}, C_{(3)}, B)}$ to the equations of motion for a D -membrane follows immediately.

Remark 6.2.4.1. [contribution only to first-order terms in EOM] As observed from these examples, for lower dimensional D-branes, the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C, B)}$ in the action contributes an additional set of *first-order* nonlinear differential-expression terms to the system of equations of motion for D-branes. In particular, they preserve the signature of the original system from the Dirac-Born-Infeld term $S_{DBI}^{(\Phi, g, B)}$ in the action.

The current notes D(13.1) lay down some foundation toward the dynamics of D-branes along the line of our D-project. Solutions to the system of equations of motion from the total action $S_{DBI}^{(\Phi, g, B)}(\varphi, \nabla) + S_{CS/WZ}^{(C, B)}(\varphi, \nabla)$ for a D-brane world-volume should be thought of as an Azumaya/matrix version of minimal submanifolds or harmonic maps, twisted/bent, on one hand, by the (dynamical) gauge field ∇ on the domain manifold X with a (noncommutative) endomorphism/matrix function-ring and, on the other hand, by the background field (Φ, g, B, C) , created by closed (super)strings, on the target space(-time) Y . Further details, issues, and examples are the focus of the sequels.

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chienhao.liu@gmail.com, chienliu@math.harvard.edu;
yau@math.harvard.edu